Technical Report

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Optimum Shock Isolation

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13. ABSTRACT (Maximum 200 words)

This report includes a discussion of the most significant achievements developed in Russia and the former Soviet Union in the theory of optimal shock isolation. Various mathematical models describing the behavior of an object mounted on a moving base which is subjected to a shock disturbance are presented. Several types of performance criteria for isolation are considered, the most important of which are the peak force transmitted to the body to be isolated and the maximum displacement of the body relative to the base.

Basic problems of shock isolation are defined and the methods for solving these problems are described. Particular attention is paid to the limiting performance problem, the solution of which provides an understanding of the physical limits for the isolation efficiency. From mathematics point of view, the limiting performance problem is a special kind of open-loop optimal control problem, and the methods of optimal control, which currently are very well developed, can be applied to the limiting performance analysis. Methods for the synthesis of parameters for specific isolator configurations (feedback control laws) for shock isolators are described. The theoretical results are supported by numerous examples of solved problems for the design of optimal shock isolators. Several specific designs are presented in-depth.

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PREFACE

P.1 HISTORICAL PERSPECTIVE.

From the modern engineering point of view impact, shock, and vibration isolators are control systems that react to the dynamic excitation (disturbance) of the object to be protected so as to mitigate undesirable effects of shock and vibration. Therefore, the theory of shock and vibration isolation can be considered as a special case of the general theory of control. Modern control theory can be considered to be composed of two areas: the theory of automatic regulation and the theory of optimal control. The theory of automatic regulation traditionally deals with the synthesis of the feedback control laws providing prescribed qualitative properties for the controlled systems. These properties are mostly associated with the stability of certain motions of the system, which the controller is attempting to maintain. The subject matter of the theory of optimal control is the construction of the control laws (either open-loop or feedback) which provide the best operating mode for the controlled system. To single out the best operating mode, a performance index (optimization criterion) is introduced that quantitatively evaluates the quality of the control process. Pontryagin and Bellman laid the foundation for the modern theory of optimal control. Pontryagin with his colleagues have formulated and proved the *maximum principle*. This principle provides the necessary optimality conditions for a wide class of optimal control problems and gives an efficient mathematical tool for the practical calculation of open-loop controls. Bellman has formulated the *principle of optimality* and suggested the method of *dynamic programming* which allows the construction of a field of open-loop optimal trajectories and calculation of an optimal control in feedback form. For details, see the classic books by Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko (1962) and Bellman (1957). The fundamentals of the optimal control theory are expounded in numerous textbooks, for instance, in Bryson and Ho (1975), Lee and Markus (1967), and Leitmann (1981).

Several treatises have been published on the theory of shock and vibration isolation viewed as control systems. In the book by Kolovskii (1976) the behavior of isolation systems is investigated by the methods of nonlinear mechanics. The book provides a classification of shock and vibration absorbers, according to their intended use and mechanical structure, as well as a classification of common types of disturbances. A detailed presentation of various mathematical methods to analyze the dynamics of single- and multi-degree-of-freedom systems for different types of excitations is given. Frolov and Furman (1990) investigate the dynamics of objects isolated from vibration and calculate the *isolator characteristics* (control laws) using the techniques of vibration theory. Mechanical properties of the isolation systems which are in most common use are analyzed. Considerable attention is paid to the analysis of isolators with hydraulic devices. In the book by Kolovskii, the isolation systems are treated from the viewpoint of the automatic regulation theory. A comprehensive presentation of modern approaches to the protection of objects from shock and vibration loads can be found in the handbook *Engineering Vibrations* (Frolov, 1995).

P.2 OPTIMIZATION OF CHARACTERISTICS OF SHOCK AND VIBRATION ISOLATORS.

Rather topical in modern engineering are problems of protection of occupants and payloads of various transport vehicles from intensive shock and vibration loads. Such loads can occur, for example, in the motion of an automobile along an uneven road at high speed or during the landing of an aircraft.

Shock loads are particularly high in crash situations. In this case, survivability of the occupants depends on the efficiency of shock isolators with which the vehicle is equipped. The need to have isolation systems that protect occupants and equipment from extremely high shock and vibration loading, on the one hand, and advances in the development of the mathematical theory of optimization, on the other hand, have stimulated the appearance of numerous publications on the optimization of shock and vibration isolation systems.

P.3 SYSTEMS WITH SMALL NUMBER OF DEGREES OF FREEDOM.

Much of the literature deals with a single-degree-of-freedom system. In this case, the object to be protected (*the body being isolated)* and the movable *base*, on which the object is placed, are considered as rigid bodies. The object is attached to the base by an isolator. It is assumed that the base translates along a straight line and the object being isolated can move relative to the base along the same line. The consideration of such simplified systems is advisable for two reasons. First, such system models satisfactorily describe the behavior of many real systems. Second, relative simplicity of this system model makes it possible to carry out a complete analysis and to obtain readily interpretable results. Often the simple system can be the basis of an investigation of the behavior of more complicated systems.

Let M and m be the masses of the base and the body being isolated, respectively. A force $\sigma(t)$ specified as a function of time is applied to the base. The isolator introduced between the body to be protected and the base generates the force $g(x, \dot{x}, t)$, which is applied to the body and depends on the displacement x of the body relative to the base, its relative velocity $\dot{x}$, and time t. According to Newton's third law, the force $-g$ is applied by the isolator to the base. The motion of the system is governed by the set of equations

$$M\ddot{z} + m(\ddot{x} + \ddot{z}) = \sigma(t), \qquad m(\ddot{x} + \ddot{z}) = g(x, \dot{x}, t) \tag{P.1}$$

where z is the displacement of the body with respect to a fixed (inertial) reference frame.

To obtain the equation of motion of the body being isolated, eliminate the variable $\ddot{z}$ from Eq. (P.1). This yields

$$\ddot{x} - \frac{g(x, \dot{x}, t)}{\mu} = -\frac{\sigma(t)}{M}, \quad \mu = \frac{Mm}{M+m}. \tag{P.2}$$

The quantity μ is called the *reduced mass* of the system of two bodies. If the motion $z(t)$ of the base, rather than the force applied to it, is prescribed, the relative motion of the object being isolated is governed by the equation

$$\ddot{x} - \frac{g(x, \dot{x}, t)}{m} = -\ddot{z}(t). \tag{P.3}$$

In the theory of isolation systems, the distinction is made between the *kinematic* and *dynamic* disturbances (excitations) of the system. The excitation is dynamic if the force applied to the base is prescribed and kinematic if the acceleration of the base is prescribed. Thus, Eq. (2) governs the motion of the body being isolated in the case of the dynamic excitation, and Eq. (3) corresponds to the kinematic excitation. Equations (2) and (3) can be represented in the unified form

$$\ddot{x} + u(x, \dot{x}, t) = F(t), \tag{P.4}$$

where $u = -g/\mu$ and $F = \sigma/\mu$ for the case of dynamic excitation; for the case of kinematic excitation, $u = -g/m$ and $F = -\ddot{z}$. The control variable $u(x, \dot{x}, t)$, just as the force $g(x, \dot{x}, t)$, will be referred to as the *characteristic of the isolator.*

Introduce the two *performance criteria*

$$J_1 = \max_t |x(t)|, \tag{P.5}$$

$$J_2 = \max_t |u(x(t), \dot{x}(t), t)|. \tag{P.6}$$

The performance criterion J_1 defines the peak relative displacement of the body being isolated, and the performance index J_2 defines the peak absolute acceleration of the body, which is proportional to the force transmitted to the body by the isolator.

Consider two optimization problems:

P.3.1 Problem 1. For the system governed by Eq. (P.4) subject to the initial conditions $x(0) = 0,\ \dot{x}(0) = 0$, find the control function u, from a specified class Y of admissible controls, that minimizes the peak displacement of Eq. (P.5), provided the constraint

$$|u| \leq U,$$

where U is a prescribed constant, is satisfied.

Note that this constraint is equivalent to $J_2 \leq U$.

P.3.2 Problem 2. For the system governed by Eq. (P.4) with the initial conditions $x(0) = 0,\ \dot{x}(0) = 0$, find the control function u, from a specified class Y of admissible controls, that minimizes the peak acceleration of Eq. (6), provided the constraint on the peak displacement

$$J_1 \leq D,$$

where D is a prescribed constant, is satisfied.

Problems 1 and 2 are simple but, at the same time, very important problems of the theory of shock and vibration isolation. Of particular interest is the *problem of limiting isolation capabilities (limiting performance problem).* In a limiting performance analysis, the isolator function u is sought as a function of time alone. That is, no state-space configuration is prescribed for the isolator and the solution of Problems 1 or 2 gives the lower bound for the criterion to be minimized. The result is the absolute optimal performance, or, as mentioned, the "limiting isolation capabilities" or the "limiting performance". In control terminology, the problems are approached from the standpoint of open-loop control rather than as feedback control problems. It would appear that the limiting performance problems were first defined by Sevin in the late 1950s. Limiting performance solutions are considered in numerous publications, e.g., in that by Guretskii (1965a) who considers the case where the criterion to be minimized is the peak displacement J_1.

It is established that the optimal isolator characteristic is piecewise constant and can always be constructed so that it assumes the values $+U$, $-U$, or $F(t)$. With allowance for this property, a graphical-analytical method for the construction of the optimal control was suggested by Sevin

and by Guretskii (1969a). In Guretskii (1968), estimated values were given for the maximum number of switching points, depending on the number of the time intervals where the absolute value $|F(t)|$ of the disturbance exceeds U.

In Guretskii (1965b), estimates of the minimal and maximal displacements of the body being isolated are made. For the cases, where the disturbance has the form of a rectangular pulse, sine wave, or a cosine quarter-wave, analytical expressions are obtained that permit one to ascertain the limiting capabilities of protection.

Guretskii, Kolovskii, and Mazin (1970) provide limiting isolation capabilities as well as a solution of the problem of synthesis of an isolator that consists of an elastic element and a damper. The synthesis is based on the time-history of the optimal control.

Saranchuk and Troitskii (1969) solved the problem of minimization of the peak displacement of a single-degree-of-freedom system subjected to a periodic disturbance. Only the steady-state motion with a period equal to that of the excitation is considered.

Manoilenko and Rutman (1974) investigated the limiting isolation capabilities for Problem 2, in which the peak acceleration is minimized. They used an "elastic analogy". The plot of the time history of the double integral with respect to time of the absolute acceleration of the object to be protected is compared to the equilibrium configuration of an elastic band. The integral approximation of the peak value of the acceleration magnitude is associated with the strain energy of the band.

Sevin and Pilkey (1971) prepared a brief treatise containing limiting performance problem definitions and several solution techniques.

P.4 OPTIMAL FEEDBACK CONTROL OF ISOLATORS.

The literature is abundant with descriptions of attempts to select optimal parameters of isolators with given configurations, i.e., feedback systems. The number of papers is so great that there is little incentive to discuss them here. However, some cases of special interest will be mentioned.

Troitskii (1967) solved the problem of synthesis of the optimal isolator feedback characteristic for a single-degree-of-freedom system. The performance index to be minimized was the peak displacement J_1. The isolator characteristic is sought as a function of the system phase coordinates (displacement and velocity) and time. The necessary optimality conditions are established. The optimal isolator feedback characteristic is constructed for the impulse excitation $F(t) = \beta\delta(t)$. Saranchuk and Troitskii (1971) investigated the problem of synthesis of the optimal isolator feedback characteristic for the case of periodic excitation.

Bolychevtsev (1971, 1973) constructed the optimal isolator feedback characteristics for the case where the external disturbance is an infinite sequence of periodically occurring instantaneous shocks. The shocks have identical intensities, either all in the same direction or in alternating directions. The optimization criterion (the performance criterion to be minimized) is the peak displacement of the body being isolated.

A problem of the optimal isolation of a multi-degree-of-freedom system is considered in Guretskii

(1965c). Optimality is defined as the minimization of the maximum (over all coordinates) of the peak displacements in each of the coordinates.

Often in design it is required that the isolator characteristic depend only on the state variables x and $\dot{x}$ and be independent of time. If the class of admissible characteristics is rather wide (for example, all piecewise continuous functions $u(x, \dot{x})$ whose absolute values do not exceed a prescribed level), then the choice of the optimal characteristic is a very complicated problem. A reasonable approach in this case is to seek the optimal characteristic among a parametric family $u(x, \dot{x}, a_1, \ldots, a_n)$, thereby reducing the original problem of optimal control to the minimization of a function of many variables. From the engineering point of view, this means that the structural schematic or configuration of the isolator has been determined and it remains to choose the isolator design variables $a_1, \ldots, a_n$ in an optimal way. Such an approach has been described in Guretskii (1966a) and Schmidt and Fox (1964) and turned out to be rather fruitful in solving practical problems. Using this approach, Guretskii (1966b) determined optimal isolator characteristics for a single-degree-of-freedom system excited by a rectangular pulse. The design variables (stiffness and damping coefficients) were found for the linear undamped isolator, linear damped isolator, and the isolator with a Coulomb characteristic.

Bolychevtsev, Zhiyanov, and Lavrovskii (1975) found the optimal design variables of the linear isolator characteristic $u(x, \dot{x}, a_1, a_2) = a_1 x + a_2 \dot{x}$ providing the minimum amplitude of the steady-state solution of Eq. (P.4). The problem is solved for an infinite sequence of periodically occurring impulsive shocks which act with the same intensity in a single direction.

The experience of solving optimal shock and vibration isolation problems shows that in many cases, commonly used isolators can provide the isolation performance close to the limiting one if the isolator design variables are chosen optimally. Bolychevtsev and Borisov (1976) solved the problem similar to that considered in Bolychevtsev (1971), the only difference being that the optimal isolator characteristic is sought among the two-parameter family of linear functions $a_1 x + a_2 \dot{x}$. It is shown that if the force allowed to be transmitted to the object being isolated is sufficiently large, the linear isolator with optimal parameters can provide the protection quality close to the limiting performance. The technical implementation of the linear isolator is much simpler than that of the optimal isolator constructed by Bolychevtsev (1971).

The parametric optimization technique has been used to find near-optimal isolator characteristics by many other authors, among them are Afimiwala and Mayne (1974), Bartel and Krauter (1971), Karnopp and Trikha (1969), Kwak, Arora, and Haug (1975), and Wilmert and Fox (1972).

P.5 MULTI-CRITERIA OPTIMIZATION.

The design of isolation systems with several performance criteria to be optimized, requires a multicriteria procedure. There are several approaches to the choice of the design variables of such systems. The most common approach involves the optimization with respect to one of the performance criteria, while the other criteria are constrained. The constraints are imposed so as to keep the responses corresponding to the constrained criteria within admissible limits. This approach was discussed above for the case of two performance criteria (the peak relative displacement and the peak absolute acceleration of the body being isolated).

Another approach involves the optimization with respect to a combined functional assembled from the original performance criteria, with weighting coefficients. As a rule, the combined functional is a linear combination of the original criteria. This approach, for example, was used by Karnopp and Trikha (1969) to choose the design variables of shock and vibration isolators.

Bolychevtsev and Lavrovskii (1977) suggested that a Pareto-optimal set be constructed in the space of the design variables of an isolation system to be designed. The Pareto-optimal set possesses the property that for any point of the set, there is no other point at which all performance criteria would be simultaneously improved. In other words, the Pareto-optimal set is a set of trade-off values of the design variables. Bolychevtsev and Lavrovskii apply the approach associated with the construction of the Pareto-optimal set to analyze the isolation system of the model of a walking machine with two degrees of freedom and three performance criteria. An effective method for constructing the Pareto-optimal set is described for the case of two design variables. The method involves the analysis of the level curves of the performance criteria, considered as functions of the design variables, and successive "cutting off" of nonoptimal portions of the design variable admissible domain. The approach for the calculation of isolator characteristics involving the construction of the Pareto-optimal set was also used by Rao and Hati (1980), Balandin and Markov (1986), and Statnikov and Matusov (1995).

P.6 EFFICIENCY OF SHOCK AND VIBRATION ISOLATION.

Ishlinskii (1963, 1987) showed that the isolation of external disturbances applied to a base is efficient only if the acceleration (deceleration) path of the base, i.e., displacement of the base during the accelerated (decelerated) motion, does not exceed the peak relative displacement of the body being isolated. Practically, this means that the isolation provides effective protection only from impact (shock) disturbances or from high-frequency vibrations. Impact disturbances are characterized by high intensity and short duration. If the impact duration is so short that during the impact time the base moves through a distance which is much less than the rattlespace (characteristic dimension within which the body being isolated is allowed to move with respect to the base), then the isolator can be designed so that the peak displacement of the body being isolated is considerably larger than the acceleration path of the base. Vibration is characterized by long-term disturbances (the forces applied to the base for dynamic disturbances or the accelerations of the base for the kinematic disturbances) that are periodic or near-periodic, changing in magnitude and direction. If the vibration frequency is high, then the time interval during which the acceleration of the base does not change direction is small, and the distance covered by the base during this time (the acceleration path) is also small. If the acceleration path is much less than the rattlespace, then, just as in the case of impact, the peak relative displacement of the body being isolated is considerably larger than the acceleration path.

The operating quality of shock isolators is usually described in terms of certain characteristics of the transient motion of the body being isolated, whereas the quality of vibration isolators is determined by the characteristics of steady-state forced oscillations. This distinction means that shock isolation problems are treated differently from vibration isolation problems. There are numerous publications on these problems, some of which will be cited here. Problems of optimal shock isolation were investigated by Afimiwala and Mayne (1974), Babitskii and Izrailovich (1968), Balandin (1985, 1988, 1989a, 1989b), Balandin and Malov (1987), Balandin and Markov (1986), Bartel and Krauter (1971), Bolotnik (1974, 1975, 1977, 1983, 1993), Bolotnik and

Kaplunov (1980), Bolychevtsev (1971, 1973), Bolychevtsev and Borisov (1976), Bolychevtsev, Zhiyanov, and Lavrovskii (1975), Eliseev and Malinin (1990), Guretskii (1965b, 1966b), Guretskii, Kolovskii, and Mazin (1970), Karnopp and Trikha (1969), Kononenko and Podchasov (1973), Ruzicka (1970a, 1970b), Ryaboy (1996), Schmidt and Fox (1964), Sevin (1972), Sevin and Pilkey (1971), Troitskii (1967), and Wilmert and Fox (1972). Various problems of optimal vibration isolation were considered by Akulenko and Bolotnik (1979), Akulenko, Bolotnik, and Kaplunov (1982), Bolotin (1969, 1970), Furunzhiev (1977), Guretskii (1969b), Guretskii and Mazin (1976), Haug and Arora (1979), Kolovskii (1976), Maksimovich (1970a, 1970b), Ryaboy (1980, 1982, 1993a, 1993b, 1994, 1995), Saranchuk and Troitskii (1969, 1971), Sevin and Pilkey (1971), and Wang and Pilkey (1975).

P.7 OPTIMAL DESIGN OF ISOLATORS FOR A CLASS OF EXTERNAL DISTURBANCES.

In the publications cited above, the external disturbance function $F(t)$ in Eq. (4), was assumed to be prescribed. However, frequently in practice, information about the external disturbance is incomplete, and it is reasonable to design a shock or vibration isolation system for a class of disturbances. As indicated in Sevin and Pilkey (1971), this problem has been of concern for considerable time. Saranchuk (1974) treated the problem in a minmax (game theory) setting. With this approach the optimal isolator characteristic is sought that provides the minimum value of the optimization criterion (performance index, objective function), e.g., the peak relative displacement, for the worst disturbance $F(t)$ belonging to a specified class of functions and maximizing the optimization criterion. As a rule, the optimal isolator characteristic is sought under constraints imposed on the motion of the system . In the case of Saranchuk, the objective function was the peak relative displacement, with the absolute acceleration constrained. The problem was solved for a class of periodic disturbances and for a class of disturbances which are identically zero outside a prescribed time interval.

Bolotnik (1976) applied the game theory approach to solve optimal isolation problems for the class of external disturbances $F(t)$ satisfying the integral constraint $\int_0^\infty |F(t)|dt \leq \beta_0$, where β_0 is a specified constant. This integral constraint defines a reasonably general class of shock disturbances, including impulsive impacts $F(t) = \beta\delta(t)$, if $|\beta| \leq \beta_0$, where $\delta(t)$ is the Dirac delta function. Bolotnik (1976) also investigated the optimal isolation problem for the case where the external disturbance is a series of instantaneous impacts. Unlike Bolychevtsev (1971, 1973) and Bolychevtsev, Zhiyanov, and Lavrovskii (1975), neither the intensities of the impacts, nor their directions are prescribed in advance. Constraints are imposed that restrict the maximum allowable intensity of each individual impact and the minimum time interval between successive impacts.

Balandin (1989a) investigated the problem of optimization of the design variables of an isolator consisting of a damper with a linear characteristic, a damper with the Coulomb characteristic, a nonlinear spring with a continuous characteristic, and a bang-bang spring for the class of external disturbances with the integral constraint of Bolotnik (1976). He established that for the isolator under consideration, the worst disturbance is the instantaneous impact with the maximum allowable intensity, irrespective of the values of the design variables i.e., stiffness and damping coefficients. On the other hand, this was shown not to be the case for isolators with arbitrary characteristics. For example, for a system with an isolator consisting of a linear spring and a quadratic-law damper, an instantaneous impact is not the worst disturbance. The game approach to the optimal design of shock and vibration isolators is also discussed in the book by Sevin and

Pilkey (1971). Therein, it is shown that computational techniques permit the study of optimal isolation systems with broad definitions of the external disturbance, for example, if the disturbance lies in a corridor prescribed as a function of time.

P.8 GENERALIZATIONS TO MOTIONS CONTAINING ROTATIONAL COMPONENTS.

In the majority of the works cited above, it is assumed that the base and the body to be isolated move translationally along a single direction. However, in many practical cases, the motion of the system is more complex and contains not only a translational component but also a rotational one. In this case, the number of degrees of freedom of the system increases and the form of the performance criteria becomes more complicated. If the motion of the system has a rotational component, terms describing the centripetal acceleration appear in the expression for the absolute acceleration and, moreover, the acceleration becomes dependent on the location on the body being isolated.

A problem of the optimum shock isolation of a body rotating about a fixed axis was solved by Bolotnik (1977) and Bolotnik and Kaplunov (1980). The performance criterion to be minimized was the total acceleration at a given location on the body to be isolated, while a constraint was imposed on the peak absolute value of the angle of rotation of the body. The Bolotnik (1977) paper deals with the optimization of the design variables of the isolator consisting of a spring with a linear characteristic and a damper with a linear or quadratic law characteristic. In the other paper, the limiting performance analysis is carried out. The results of the optimal shock isolation of a rotating body are compared with the corresponding results for a translating body.

Kulagin and Prourzin (1985) and Prourzin (1988) generalize the results to cases where the base and a single body being isolated perform more complex motion containing both translational and rotational components.

P.9 COMPUTATIONAL METHODS.

Even in solving relatively simple problems of optimal shock or vibration isolation for single-degree-of-freedom systems it is often necessary to utilize numerical methods which need computer implementation. The difficulties in obtaining closed-form solutions are associated primarily with nonlinearities of the equations of motion and also with the form of performance indices characteristic of the optimization of shock or vibration isolators. Often, the performance criteria are represented as a maximum of a quantity representing the isolation efficiency. For example, typical criteria are the peak displacement relative to the base or the peak absolute acceleration of the body to be isolated. These difficulties become even more significant for systems with many degrees of freedom. In this case, the development of numerical methods is rather essential and topical.

Problems of optimization of shock or vibration isolator characteristics belong to a particular class of the general optimal control problem. Numerous numerical methods for solving various optimal control problems have been developed and tested. Basic numerical methods of optimal control are presented, for example, in such books as those by Chernousko and Banichuk (1973), Fedorenko (1978), and Moiseev (1975). Many of the methods, with modifications allowing for specific features of optimum shock and vibration isolation problems, can be applied to calculate optimal

controls for isolation systems.

A complicating feature of many problems of optimal shock and vibration isolation is that the performance indices of isolation in these problems are represented as a nonadditive functional of the form

$$J(\mathbf{u}) = \max_{t\in[t_0,T]} \Phi(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), t), t) \tag{P.7}$$

where $\mathbf{x}$ is the phase vector of the system, $\mathbf{u}$ is the control vector function, $\Phi(\mathbf{x}, \mathbf{u}, t)$ is a prescribed function, and t_0 and T are, respectively, the initial and the terminal instants of the motion. Sometimes, functionals of the form of Eq. (P.7) are referred to as *maximum-type functionals*. The functionals J_1 and J_2 of Eqs. (P.5) and (P.6) are particular cases of the functional $J(\mathbf{u})$ of Eq. (P.7). Optimal control problems with functionals of such a type cannot be tackled with the traditional optimal control techniques. Hence, specific approaches to these problems were developed.

There are a number of methods for the numerical solution of optimal control problems with maximum-type functionals. One of the simplest approaches involves partitioning the time interval $[t_0, T]$ into subintervals and setting the control function to be constant or linearly varying on each of the subintervals. In such a way, the original optimal control problem can be reduced to the minmax problem for a function of a finite number of variables, the maximum being taken over all discretization points and the minimum over the values of the control variables on the subintervals. The calculation of a trial value of the function, for which the minmax is to be found, may require numerically integrating the equations of motion for a specified set of the control variable values on the discretization subintervals. The theory of the minmax of functions of a finite number of variables and the numerical algorithms based on this theory are presented in the book by Demyanov and Malozemov (1972). One of the approaches in question was applied to the limiting performance analysis of shock and vibration isolation systems by Vinogradova (1974). The attractive feature of the methods based on the discretization of the control function is their simplicity. However, these methods are complicated because in many practical cases, the discrete mesh must be very fine to provide a high precision for the solution, which can lead to computational challenges.

Another approach involves an appropriate approximation of the maximum-type functional by an additive functional (for example, integral or terminal functional) which can be minimized by the traditional optimal control methods. For the typical case, where $\Phi(\mathbf{x}, \mathbf{u}, t) > 0$, the maximum-type functional $J(\mathbf{u})$ of Eq. (P.7) is often replaced by the integral

$$J_\nu(\mathbf{u}) = \int_{t_0}^{T} \Phi^\nu(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), t), t)dt, \tag{P.8}$$

where ν is a sufficiently large positive number. This replacement is based on the well-known relation

$$\lim_{\nu\to\infty} [J_\nu(\mathbf{u})]^{1/\nu} = J(\mathbf{u}) = \max_{t\in[t_0,T]} \Phi(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), t), t), \tag{P.9}$$

which is valid under rather general conditions.

Sometimes, if the function Φ is independent of $\mathbf{u}$, the maximum-type functional is replaced by a terminal functional with an unknown time for the termination of the process. This approach was

used, for example, by Kuznetsov and Chernousko (1968) and Troitskii (1967) to solve some problems of optimal control in mechanical systems whose performances were evaluated by maximum-type functionals. The condition of the vanishing of the total derivative of the function $\Phi(\mathbf{x}(t), t)$ with respect to time is adopted as the condition of the process termination. At this instant, the necessary condition for the extremum of the function $\Phi(\mathbf{x}(t), t)$ is satisfied. This approach can be effective only for the rare cases where the first local extremum of the function $\Phi(\mathbf{x}(t), t)$ is the global maximum of this function on the interval $[t_0, T]$ for any control $\mathbf{u}$. In the other cases, the use of this method is problematic.

Because of the practical importance of optimal control problems with maximum-type functionals (in particular, in connection with optimal shock and vibration isolation), computational methods have been developed that take into account the special mathematical features of these problems. Viktorov and Larin (1969) suggested a computational algorithm which is a modification of the method of gradient descent in the space of control functions (Shatrovskii, 1962) for maximum-type functionals. This method was applied to solve the optimal isolation problem posed in Guretskii (1965a) for a single-degree-of-freedom system for two special kinds of external disturbances.

A number of methods for numerical solution of optimal control problems with the functional of the form $\max_t \Phi(\mathbf{x}(t), t)$ were developed by Silina (1976) and Timoshina and Shablinskaya (1980). These methods are based on the necessary optimality conditions. It is assumed that the function $\Phi(\mathbf{x}(t), t)$ has a finite number of points of local extrema for any admissible control. This property permits the reduction of the original problem of optimal control to the search for the extremum in a finite-dimensional space, which is simpler computationally. The methods developed were applied to the limiting performance analysis of shock isolation systems.

Sevin and Pilkey (1967a, 1971) used the dynamic programming technique to solve control problems with maximum-type functionals for single-degree-of-freedom systems. In Sevin and Pilkey (1967a), the worst disturbance problem is solved for a body attached to a base by a linear isolator. The class of admissible disturbances is defined as the class of functions of time with a prescribed integral. In Sevin and Pilkey (1971), dynamic programming is applied to solve the problem of limiting isolation capabilities for a system subject to completely or incompletely prescribed disturbances. In both cases, the peak relative displacement of the body being isolated was chosen to be the performance index.

Wang and Pilkey (1975) suggested a method for approximate solution of the limiting performance problem for linear multi-body systems subjected to periodic disturbances. For steady-state motions the control is sought in the form of a truncated Fourier series in terms of harmonics whose frequencies are multiples of those of the external disturbance. The Fourier coefficients are determined by numerically solving a nonlinear programming problem so as to provide the minimum for the performance criterion. Even more significant is the use of the Fourier series approach for problems of optimal isolation of linear systems subject to transient excitations.. This leads to a linear programming problem in which the coefficients of the Fourier series are the unknowns. In recent years this has become a very viable method since the number of unknowns is less than for the approach discussed earlier in which the isolator force is discretized as a piecewise constant function.

In Larin (1969), a numerical algorithm for optimization of the design variables of

single-degree-of-freedom isolation systems is presented. The performance criterion to be minimized is the peak displacement of the body being isolated. The method is based on the reduction of the original problem to a nonlinear programming problem which is solved by the gradient descent in the design variable space. On each iteration, to calculate the objective function it is necessary to integrate the equation of motion and to calculate the peak displacement of the body.

Hsiao, Haug, and Arora (1979) developed a numerical method for the optimization of dynamical systems with many degrees of freedom for the case where the criterion to be minimized has the form of a maximum of a function of the system phase variables. The approach involves an equivalent replacement of the maximum-type functional by an integral functional and application of the Lagrange multiplier technique. The method was used to calculate the optimal design variables of spring-and-damper isolators for systems with one and two degrees of freedom.

Sevin and Pilkey (1971) suggested a numerical method for determining the optimal design variables of shock and vibration isolation systems which does not require integrating the equations of motion on each iteration but needs the limiting performance problem to be solved beforehand. The heuristic basis of the method is the contention that the characteristics of a well designed system must be close to those providing the limiting performance. Hence, in this approach, the feedback control parameters are chosen such that the time history of the feedback control force is as close as possible to the open loop optimal control corresponding to the limiting performance. The practical application of this method seems to be limited. However, it can be used for preliminary testing calculations, for example, with the aim of finding out whether a selected design of the isolation system allows adjusting the design variables so as to provide the limiting performance characteristics for the shock or vibration isolation system.

P.10 MONOGRAPHS ON THE OPTIMAL SHOCK AND VIBRATION ISOLATION.

There are several monographs devoted to optimal shock and vibration isolation. A fundamental book in this field was written by Sevin and Pilkey (1971). This book presents general mathematical statements of basic problems of optimal shock and vibration isolation, describes typical performance criteria and constraints imposed on the performance characteristics, control variables, and design variables of isolation systems, and discusses the adequacy of the mathematical description relative to the properties of real isolation systems used in engineering. A classification of external disturbances and types of isolators is given. The problem of the optimal isolation of shock and vibration is stated in a general form for a system with an arbitrary number of degrees of freedom. The limiting performance problem and the problem of parametric optimization are presented as practically important, distinct, special cases of the general optimization problem. A linear programming solution is outlined for the limiting performance problem of multi-degree-of-freedom linear systems. Particular attention is paid to single-degree-of-freedom systems. For such systems, it is proved that the optimal open-loop control providing the limiting performance for the isolation system can be constructed in the form of a piecewise constant function of time that takes either upper or lower boundary values. A graphical-analytical method to construct the optimal isolator characteristic (optimal control) for the limiting performance problem is suggested. In major features, this method is similar to the method of Guretskii (1969a). Various numerical methods for constructing the optimal isolator characteristics or determining the optimal design variables for isolation systems are discussed. In particular, linear programming (for linear systems) and dynamic programming (for linear and

nonlinear systems) solutions for the limiting performance problem are developed. The book is richly illustrated with numerical examples, graphs, and figures. An annotated bibliography of works dealing with optimum shock and vibration isolation is presented at the end of the monograph.

The monograph by Furunzhiev (1971) is devoted mostly to stochastic problems of optimal vibration isolation of systems subjected to random disturbances. The author discusses mathematical models of systems with stochastically characterized performance criteria. Various problems of optimal control and parametric optimization of vibration isolators are stated and methods for their solution are given. The book also contains descriptions of computational techniques for the statistical analysis of vibration isolation systems with the aid of a computer. Numerical examples for the calculation of vibration isolation systems for transport vehicles are presented.

The book by Kolovskii (1976) contains various statements of optimal shock and vibration isolation problems and discusses different approaches to their solution. In particular, the problem of the limiting isolation capabilities (limiting performance problem) is formulated. The graphical-analytical method of Guretskii (1969a) and various numerical methods to solve the limiting performance problem for single-degree-of-freedom isolation systems are presented. A comparative analysis of these methods is given. Various approaches to the synthesis of optimal or near-optimal feedback isolator characteristics are expounded. Particular attention is paid to the optimal isolation of rigid bodies performing complex motion from shock or vibration. Mathematical models of isolation systems for rigid bodies are considered, statements of the corresponding optimization problems are presented, and methods for solving these problems are discussed. The majority of the issues discussed in the monograph by Kolovskii are illustrated with numerical examples. A bibliography of works on the optimization of shock and vibration isolation systems is presented.

In the book by Bolotnik (1983), a complete solution of the problem of the optimal isolation of a single-degree-of-freedom system from an impulsive impact is given. The performance criteria are the peak displacement of the body being isolated with respect to the base and the peak acceleration of this body with respect to a fixed reference frame. Cases where the body moves along a straight line and where it rotates about an axis are considered. The optimal performance characteristics are obtained, and feedback isolators providing the limiting performance or approximations to it are discussed. For the case of the rectilinearly moving body, all spring-and-damper isolators with power-law characteristics that provide the limiting performance for the shock isolation system are identified. The problem of optimization of the isolator characteristic for a class of external disturbances is considered in the game theory (minmax) setting. The optimal and near-optimal feedback controls are constructed for the case where the body being isolated moves rectilinearly and the integral of the absolute value of the applied force to which the system can be subjected is constrained by a prescribed quantity. A number of problems for the optimal vibration isolation of a mechanism containing an unbalanced rotor are solved.

Genkin and Ryaboy (1988) investigate multi-mass absorbers (isolators) for harmonic vibrations. The isolators consist of inertial and elastic members. The problem of minimization of the isolator mass, provided the dynamic load transmitted to the object to be protected is reduced to a

prescribed level, is solved. The reduction of the dynamic load is characterized by the transmissibility coefficient of the system. In the general case, the prescribed reduction level may depend on the vibration frequency. Estimates for the minimal mass of the isolator, depending on the maximum allowable value of the transmissibility coefficient, are given. The authors suggest a number of design schematics for optimal vibration isolators.

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SECTION 1

SYSTEMS WITH ISOLATORS AND PROBLEMS OF OPTIMIZING THEIR CHARACTERISTICS

1.1 MECHANICAL SYSTEMS WITH ISOLATORS.

1.1.1 Purpose and Effectiveness of Isolation.

The use of isolators in various machines and mechanisms is intended for reducing forces transmitted to certain components of the devices. These forces are caused by either the accelerated motion of the mechanism or dynamic disturbances applied to it.

An alternative use of isolators is to reduce displacements. For example, one of the purposes of using isolators, e.g., springs and dampers, in automobiles is the reduction in the amplitude of vibration of the body, due to irregularities of the road profile.

Consider some simple examples of typical systems equipped with isolators. A mechanical system consisting of an object (a body) mounted on a movable base can serve as a model for a wide class of engineering systems including devices on machines, ships, aircraft, spacecraft, automobiles, motorcycles, locomotives, etc. We assume that the object is a rigid body, which can move relative to the base along the line of motion of the base. The base is assumed to move along a straight line.

The system described is governed by the differential equation

$$m\ddot{x} = -m\ddot{y} + f(x, \dot{x}), \tag{1.1}$$

where m is the mass of the object, x is the object displacement relative to the base, y is the coordinate of the base in an inertial (fixed) reference frame, and f is the force of interaction between the body and the base. Dots above letters denote derivatives with respect to time t.

The force f depends on the elastic and dissipative properties of the junction between the object and the base and, as a rule, is a function of the relative displacement, x, and relative velocity, $\dot{x}$, of the object. If the object is rigidly connected to the base, i.e., the junction allows no relative displacements, then Eq. (1.1) implies $f = m\ddot{y}$ and the object is acted upon by a force equal, apart from the sign, to the inertia force due to the motion of the base. This force may be so great that the intended function of the object is impeded. To circumvent this, the rigid connection to the base is avoided in favor of an isolator, a special engineering contrivance that separates the object from the base and generates a control force so as to reduce the load on the object, compared with the rigid fastening.

Various isolators are used in engineering, from comparatively simple springs and dampers of cars and motorcycles to rather complex systems controlled by computers.

The motion of the object connected with the base by means of an isolator (Fig. 1.1) is governed by the equation

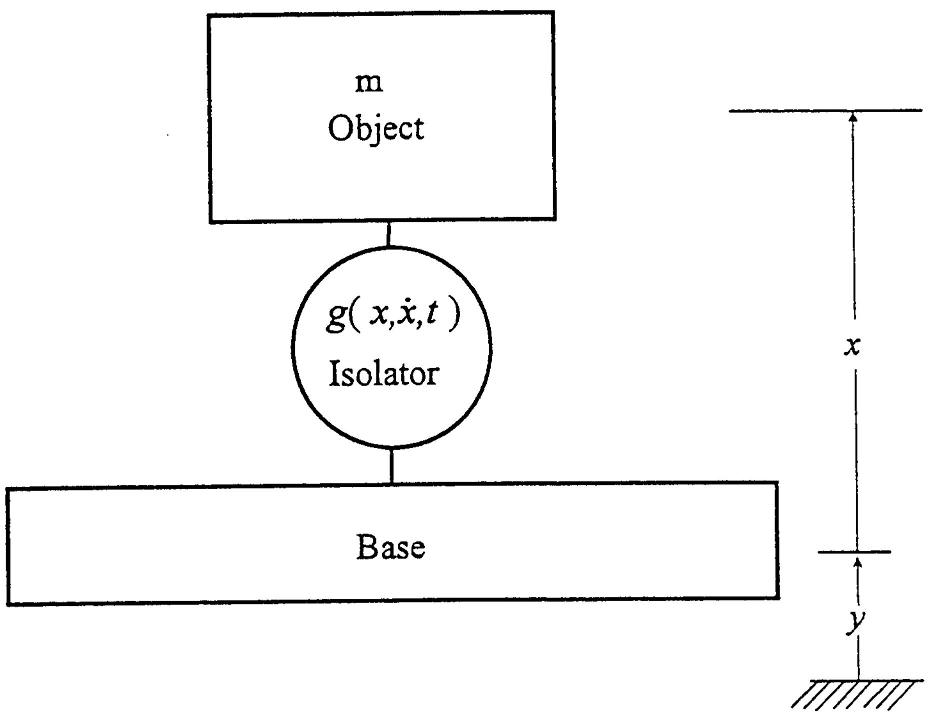


Figure 1-1. The object connected to the base by means of an isolator.

$$m\ddot{x} = -m\ddot{y} + g(x, \dot{x}, t), \tag{1.2}$$

where $g(x, \dot{x}, t)$, the *isolator characteristic*, is a control function that in the general case depends on the state variables, x and $\dot{x}$, and time t. The explicit appearance of time t allows for *active isolation systems.*

1.1.1.1 Linear Spring Isolator. In many cases, the use of even very simple isolators leads to a considerable reduction of the load on an object. Let us assume in Eq. (1.2)

$$\ddot{y} = \begin{cases} a, & \text{for } 0 \le t \le t_0; \\ 0, & \text{for } t > t_0; \end{cases} \tag{1.3}$$

$$g(x, \dot{x}, t) = -kx;$$

$$x(0) = \dot{x}(0) = 0.$$

This means that the base first moves at a constant acceleration a and at instant t_0 the base motion becomes uniform. A linear spring with stiffness factor k serves as the isolator. The object being isolated is at rest with respect to the base at the initial instant $t = 0$. The solution of Eq. (1.2) under the conditions of (1.3) is

$$x(t) = \begin{cases} -2\frac{a}{\omega^2}\sin^2\frac{\omega t}{2}, & \text{for } 0 \le t \le t_0; \\ -2\frac{a}{\omega^2}\sin\frac{\omega t_0}{2}\sin\left[\omega\left(t - \frac{t_0}{2}\right)\right], & \text{for } t > t_0, \end{cases} \tag{1.4}$$

where $\omega^2 = k/m$ is the square of the natural frequency.

It follows from (1.4) that the maximum absolute value of the force acting on the object being isolated is

$$g_{\max} = \max_t |g(x(t), \dot{x}(t), t)| = \max_t |kx(t)| = \begin{cases} 2ma\sin\frac{\omega t_0}{2}, & \text{for } t_0 \le \frac{\pi}{\omega} \\ 2ma, & \text{for } t_0 \ge \frac{\pi}{\omega} \end{cases}. \tag{1.5}$$

For the object rigidly connected to the base, the maximum absolute value of the force acting on the object is $\tilde{g}_{\max} = ma$.

It follows from (1.5) that if the duration of the accelerated motion of the base is comparatively short ($t_0 < \pi/(3\omega)$), then $g_{\max} < ma = \tilde{g}_{\max}$ and the spring isolator provides a reduction of the maximum load on the object. Quantatively, the isolation efficiency can be evaluated by the ratio

$$\frac{\tilde{g}_{\max}}{g_{\max}} = \begin{cases} [2\sin\frac{\omega t_0}{2}]^{-1}, & \text{for } t_0 \le \frac{\pi}{\omega} \\ 1/2, & \text{for } t_0 \ge \frac{\pi}{\omega} \end{cases}. \tag{1.6}$$

The greater the ratio of (1.6), the more effective the isolation. The isolation efficiency $\tilde{g}_{\max}/g_{\max}$ increases as the ratio of the duration of the accelerated motion of the base (t_0) to the period of natural oscillations of the body being isolated ($T = 2\pi/\omega$) decreases.

1.1.1.2 Unbalanced Rotor. Consider a mechanism mounted on a fixed base, containing an unbalanced rotor (a flywheel) (see Fig. 1.2). Assume that the rotor and the body of the mechanism can be treated as rigid bodies and the rotation axis of the rotor is rigidly connected to the mechanism body. For the rotor rotating at a constant angular velocity Ω, the motion of the body with respect to the base is described by the differential equation

$$(m_1 + m_2)\ddot{x} + f(x, \dot{x}) = m_1\Omega^2 l \sin(\Omega t + \varphi). \tag{1.7}$$

Here, x is the displacement of the body relative to the base, m_1 is the mass of the rotor, m_2 is the mass of the body, l is the distance between the mass center of the rotor and its rotation axis, φ is the initial angle of rotation of the rotor, and $f(x, \dot{x})$ is the force applied to the base by the mechanism body. The force $f(x, \dot{x})$ characterizes the connection between the mechanism body and the base.

When the body is rigidly connected to the base, the latter is acted upon by the force (dynamic load) $\widetilde{f} = m_1\Omega^2 l \sin(\Omega t + \varphi)$ that is due to the action of the rotating flywheel on the mechanism body. The maximum absolute value of this force is equal to $\widetilde{f}_{\max} = m_1\Omega^2 l$. For high angular velocities Ω of the flywheel, the magnitude of $\widetilde{f}_{\max}$ can become so large as to cause problems. An isolation system can be introduced to reduce the dynamic load between the mechanism and the base.

If the isolation system is a spring with linear rate, then the motion of the body is governed by (1.7) with $f(x, \dot{x}) = kx$, where k is the stiffness constant. Of primary interest for mechanisms with uniformly rotating flywheels are the steady-state oscillations and the concomitant dynamic loads. The free (transient) oscillations decay with time due to the energy dissipation that always occurs in real systems.

Steady-state oscillations of the system of (1.7) with $f(x, \dot{x}) = kx$ are given by

$$x(t) = \frac{m_1 l}{m_1 + m_2} \frac{\Omega^2}{\nu^2 - \Omega^2} \sin(\Omega t + \varphi), \tag{1.8}$$

$$\nu^2 = \frac{k}{m_1 + m_2}.$$

The force transmitted to the base is

$$f = kx = \frac{m_1 l \nu^2 \Omega^2}{\nu^2 - \Omega^2} \sin(\Omega t + \varphi). \tag{1.9}$$

The maximum absolute value of this force is expressed by

$$f_{\max} = \frac{m_1 l \nu^2 \Omega^2}{|\nu^2 - \Omega^2|}. \tag{1.10}$$

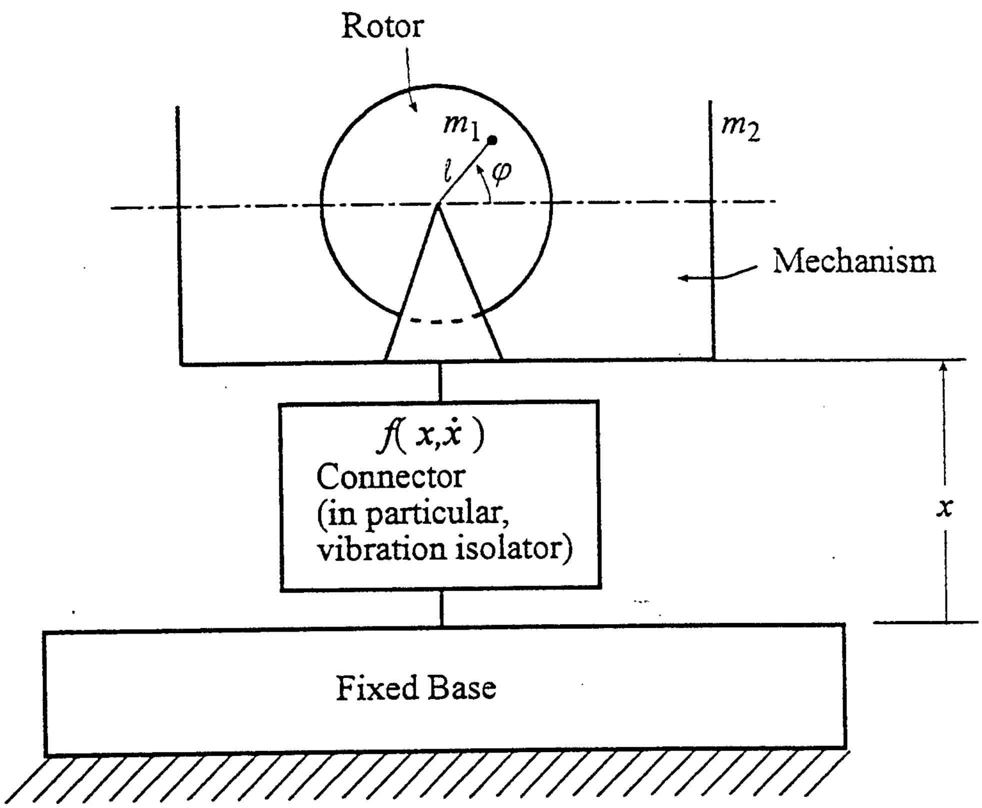


Figure 1-2. The mechanism with an unbalanced rotor on a fixed base.

Equation (1.10) implies that if $\nu^2 < \Omega^2/2$, then $f_{\max} < \bar{f}_{\max}$ and, hence, the isolation system reduces the dynamic load, relative to that for the rigid connection of the body to the base.

The use of isolators does not always lead to the desired reduction in dynamic loads. In some cases, the isolator produces the opposite effect and increases the load compared with the load level for the rigid connection to the base. For instance, it follows from Eq. (1.5) of Example 1.1 that if $t_0 > \pi/(3\omega)$, then $g_{\max} > \widetilde{g}_{\max}$, and the use of the isolator deteriorates the operating conditions of the system. It is apparent here that when the duration of the accelerated motion of the base is sufficiently long, the acceleration of the base ($\ddot{y}$) and the relative acceleration ($\ddot{x}$) of the oscillating object at some instants have the same direction. At these instants, the magnitude of the absolute acceleration ($\ddot{x} + \ddot{y}$) of the object exceeds the acceleration of the base and, hence, the force $(m\ddot{y} + m\ddot{x}) = (ma + m\ddot{x})$ acting on the body being isolated is greater than the force (ma) in the case of rigid connection of the body to the base.

For the system of Example 1.2, the use of the isolator increases the dynamic load if $\nu^2 > \Omega^2/2$.

1.1.1.3 Guidelines for the Effectiveness of an Isolator. Ishlinsky (1963, 1987) has studied the effectiveness of the isolation of objects mounted on a movable base using the model governed by Eq. (1.2). This study led him to a conclusion that the isolation is inefficient if the distance covered by the base during the accelerated motion is greater than the maximum relative displacement allowed for the body to be isolated. In what follows in this section, we will prove a proposition that substantiates this observation.

Let us call the time interval $[t_1, t_2]$ the *interval of acceleration (deceleration)* of the base if the inequality $\ddot{y}\dot{y} > 0$ $(\ddot{y}\dot{y} < 0)$ holds for any $t \in [t_1, t_2]$. By this definition, the velocity and acceleration of the base have identical (opposite) directions on the acceleration (deceleration) interval and the absolute value of the velocity increases (decreases). Refer to the distance covered by the base during the acceleration (deceleration) interval as the *acceleration (deceleration) path.*

Let us suppose that at time $t = t_1$ the object being isolated is in the state of relative rest, i.e., $x(t_1) = \dot{x}(t_1) = 0$. Consider its motion over the interval of acceleration of the base. Without loss of generality we assume $\ddot{y} > 0$ and $\dot{y} > 0$ for $t \in [t_1, t_2]$. Equation (1.2) implies that the isolation is effective, i.e., the inequality $g(x(t), \dot{x}(t), t) < m\ddot{y}(t)$ holds, on the acceleration interval only if $\ddot{x}(t) < 0$ for $t \in [t_1, t_2]$. Indeed, according to Eq. (1.2), describing the motion of the body being isolated, the force g acting on this body is expressed as $g = m(\ddot{x} + \ddot{y})$. We consider the case where the base acceleration $\ddot{y}$ does not change its direction. Without loss of generality we have assumed that $\ddot{y} > 0$. By rigidly attaching the body to be protected to the base we can always provide $g = m\ddot{y}$. Hence, to make $g < m\ddot{y}$ at all time instants we must assure that $\ddot{x} < 0$.

We assume that

(i) the function $\ddot{y}(t)$ is continuous in the interval $[t_1, t_2]$;

(ii) the isolator characteristic $g(x, \dot{x}, t)$ is continuous as a function of $x, \dot{x}$, and t;

(iii) the inequality $\ddot{x} < 0$ (the necessary condition for effectiveness of isolation) holds over the interval of acceleration of the base;

(iv) the maximum absolute value of the relative displacement of the body being isolated is bounded by a constant δ: $\max_{t\in[t_1,t_2]} |x(t)| \leq \delta$.

The last assumption is always satisfied because engineering systems are of finite size.

1.1.1.4 Proposition. Under assumptions (i)-(iv), there exist a time instant $t_* \in [t_1, t_2]$ and a constant $L > 0$ such that the estimate

$$|\ddot{x}(t_*)| \leq \frac{L\delta p}{s}, \quad p = \max_{t\in[t_1,t_2]} \ddot{y}(t) \tag{1.11}$$

occurs for any acceleration path s satisfying the inequality $s \geq \gamma > 0$, where γ is a constant.

This proposition implies that $|\ddot{x}(t_*)| \to 0$ as $s \to \infty$. Therefore, the isolation efficiency at the point t_* is reduced as the acceleration path grows.

1.1.1.5 Proof. The displacement of the object with respect to the base at the instant $t = t_2$ is represented by

$$x(t_2) = \int_{t_1}^{t_2} (t_2 - \tau)\ddot{x}(\tau)d\tau. \tag{1.12}$$

According to the mean value theorem, there exists a time instant $t_* \in [t_1, t_2]$ such that

$$x(t_2) \quad = \quad \ddot{x}(t_*) \int_{t_1}^{t_2} (t_2 - \tau)d\tau = \ddot{x}(t_*)\frac{(\Delta t)^2}{2}, \quad \Delta t = t_2 - t_1. \tag{1.13}$$

By assumption (iv), the variable x is constrained by

$$\max_{t\in[t_1,t_2]} |x(t)| \leq \delta, \tag{1.14}$$

and, hence, with allowance for Eq. (1.13), we have

$$|x(t_2)| = |\ddot{x}(t_*)|\frac{(\Delta t)^2}{2} \leq \max_{t\in[t_1,t_2]} |x| = \delta. \tag{1.15}$$

Solving this inequality for $|x(t_*)|$ gives

$$|\ddot{x}(t_*)| \leq \frac{2\delta}{(\Delta t)^2}. \tag{1.16}$$

The coordinate y of the base at the instant t_2 can be expressed as follows:

$$y(t_2) = y(t_1) + \dot{y}(t_1)(t_2 - t_1) + \int_{t_1}^{t_2} (t_2 - \tau)\ddot{y}(\tau)d\tau \tag{1.17}$$

and the acceleration path is

$$s = y(t_2) - y(t_1) = \dot{y}(t_1)(t_2 - t_1) + \int_{t_1}^{t_2} (t_2 - \tau)\ddot{y}(\tau)d\tau. \tag{1.18}$$

Since $\ddot{y} \leq p$, where p is defined in Eq. (1.11), the integral in Eq. (1.18) can be estimated as

$$\int_{t_1}^{t_2} (t_2 - \tau)\ddot{y}(\tau)d\tau \leq \int_{t_1}^{t_2} (t_2 - \tau)p d\tau = p\frac{(t_2 - t_1)^2}{2}. \tag{1.19}$$

With allowance for Eqs. (1.18) and (1.19) we obtain the estimate

$$s \leq \dot{y}(t_1)\Delta t + \frac{p(\Delta t)^2}{2}. \tag{1.20}$$

The solution for Δt of the inequality of Eq. (1.20) yields

$$\Delta t \geq \frac{[\dot{y}^2(t_1) + 2sp]^{1/2} - \dot{y}(t_1)}{p} = \left[\frac{2s}{p}\right]^{1/2} \left[\frac{\lambda - 1}{\lambda + 1}\right]^{1/2}, \tag{1.21}$$

$$\lambda^2 = \frac{\dot{y}^2(t_1) + 2sp}{\dot{y}^2(t_1)} \geq 1 + \frac{2\gamma p}{\dot{y}^2(t_1)} = \lambda_0^2.$$

Substitute Eq. (1.21) into Eq. (1.16) to obtain

$$| \ddot{x}(t_*) | \leq \frac{L\delta p}{s}, \quad L = \frac{\lambda_0 + 1}{\lambda_0 - 1}. \tag{1.22}$$

Thus we have arrived at the inequality of Eq. (1.11). This completes the proof of the proposition. A similar proposition is valid for the interval of deceleration of the base.

It follows from Eq. (1.11) that if the change of the base acceleration over the acceleration interval, i.e. $\max_{t',t'' \in [t_1,t_2]} |\ddot{y}(t') - \ddot{y}(t'')|$, is small compared with p and the maximum allowable displacement δ of the object is small compared with the path s of the base acceleration, then the isolation is ineffective because at some instants the absolute value of the relative acceleration $\ddot{x}$ of the body being isolated is small and the force acting upon it is close to the force corresponding to the rigid connection of the object to the base.

Based on this observation and on a number of concrete examples, Ishlinsky formulated the following guideline concerning the effectiveness of using isolators:

The use of isolators is effective only in cases where the path of deceleration (or acceleration) of the base does not exceed the maximum relative displacement of the object being isolated.

To illustrate this guideline, return to the system of Example 1.1. As shown previously, if the acceleration of the base is given by (1), then the spring isolator reduces the load on the object if and only if $t_0 < \pi/(3\omega)$. Assume the velocity of the base is equal to zero at time $t = 0$. Then the

base covers the distance $s = at_0^2/2$ during time t_0. From (2), the maximum absolute value of the relative displacement of the object being isolated is $\delta = (2a/\omega^2)\sin(\omega t_0/2)$. The difference $s - \delta$ is expressed by

$$s - \delta = \frac{2a}{\omega^2}h(z), \quad h(z) = z^2 - \sin z, \quad z = \frac{\omega t_0}{2}. \tag{1.23}$$

The function $h(z)$ satisfies the inequality $h(z) < 0$ in the interval $0 < z < \pi/6$, corresponding to $0 < t_0 < \pi/(3\omega)$. This follows from the fact that the function $h(z)$ is convex $(h''(z) = 2 + \sin z > 0)$ and nonpositive at the ends of the interval $[0, \pi/6]$ $(h(0) = 0, \quad h(\pi/6) = \pi^2/36 - 1/2 < 0)$. Thus, according to Eq. (1.23) we have $s - \delta < 0$ for $0 < t_0 < \pi/(3\omega)$ and, hence, the isolator is effective only if the path of the base acceleration is less than the maximum relative displacement of the object being isolated.

Consider now the case where the base performs harmonic oscillations of amplitude A and frequency Ω, i.e., $y = A\sin(\Omega t + \varphi)$, where φ is a constant. In this case, the acceleration of the base is given by $\ddot{y} = -A\Omega^2\sin(\Omega t + \varphi)$ and the motion of the object connected to the base by means of a spring of stiffness k is governed by the following differential equation:

$$m\ddot{x} + kx = mA\Omega^2\sin(\omega t + \varphi). \tag{1.24}$$

Note that Eq. (1.24) with an adjustment in the definition of the constants is the same as Eq. (1) in Example 1.2, where $f(x, \dot{x}) = kx$.

Steady-state oscillations of the system of Eq. (1.24) are given by

$$x = \frac{A\Omega^2}{\omega^2 - \Omega^2}\sin(\Omega t + \varphi), \quad \omega^2 = \frac{k}{m}. \tag{1.25}$$

The maximum relative displacement of the object being isolated is $\delta = A\Omega^2/\left|\omega^2 - \Omega^2\right|$, while the path of acceleration (or deceleration) of the base is equal to the amplitude of its oscillation, i.e., $s = A$. As was shown in Example 1.2, the isolator leads to a reduction of the maximum load on the object, compared with the value corresponding to the rigid connection of the body to the base, if and only if $\omega^2 < \Omega^2/2$. If this inequality holds, then

$$s - \delta = -\frac{A\omega^2}{|\omega^2 - \Omega^2|} < 0, \tag{1.26}$$

i.e., the isolator is effective only if the path of acceleration (or deceleration) of the base is less then the maximum relative displacement of the object being isolated.

This confirms again the applicability of the engineering guideline formulated above.

Since real structures are of finite size, the guideline permits us to conclude that isolation cannot protect objects, in the sense of reducing large forces, when the base is subjected to long duration acceleration acting in a single direction. For example, significant loads generated when launching a spacecraft cannot be mitigated by isolators. However, the use of isolators permits a significant

reduction in forces if the time of the accelerated motion of the base is short, as occurs, for example, for the vertical motion of an aircraft when landing. Also, the isolation can be effective if the acceleration changes its direction frequently, which is characteristic of high-frequency vibrations. The issues concerning the effectiveness of using isolators for vibration protection are discussed in detail by Kolovskii (1976).

1.1.2 Structure and Mathematical Models of Systems with Isolators.

By *system with isolators* we understand an aggregate of objects connected to each other by *isolators*. The objects we will call *elements* (or *members*) of the system with isolators. By isolators (isolation elements) we understand engineering devices connecting the system members so that they can move relative to each other. Isolators provide controlling interaction between the members. Machines and mechanisms used in different branches of engineering and industry, including transport vehicles (automobiles, aircraft, and locomotives), machine tools and technological equipment, household equipment (refrigerators and washing machines), etc., can be regarded as systems with isolators. The elements of an automobile regarded as a system with isolators are the body, front axle, rear axle, etc., while the isolators are spring-damper units serving to reduce the body oscillations when driving along an uneven road or devices intended for reducing the impact force transmitted to occupants during a crash. The elements of a washing machine are its body, drums, and motors, which are connected by means of spring isolators.

Often, in engineering literature, the elements of a system with isolators are classified into the base and the objects being isolated. This conventional classification is usually related to the different roles of the components of machines and mechanisms as well as to the desire to have the basic reference frame associated with a certain component of the system. For example, when investigating and designing navigation equipment for ships and aircraft, it is natural to consider the body of a ship or an aircraft as the base and to describe the motion of devices being isolated in a reference frame associated with the base. If a system with isolators contains a fixed element, e.g., the foundation on which a machine tool or another mechanism is mounted, it is usually regarded as the base and the motion of the other elements is described relative to this fixed element.

Figure 1.3 presents a schematic of a system with isolators, the elements and isolators being labeled by letters e and i, respectively.

For analysis and optimization purposes, it is important to have a mathematical model adequate to describe the behavior of a system with isolators. In most cases, it is sufficient to consider the elements of the system with isolators as rigid bodies or elastic bodies. However, more complex models may be necessary in some instances.

If the masses of the isolator parts are much less than the masses of the base and objects being isolated, then the isolators can be modeled as massless units generating control forces which depend on relative displacements and velocities of the elements of the system with isolators, as well as on time, if the isolation system is active.

The mathematical model of the system with isolators is a set of differential equations (ordinary or partial) with corresponding initial and boundary conditions. Some examples of systems with isolators governed by ordinary differential equations have already been considered in this section.

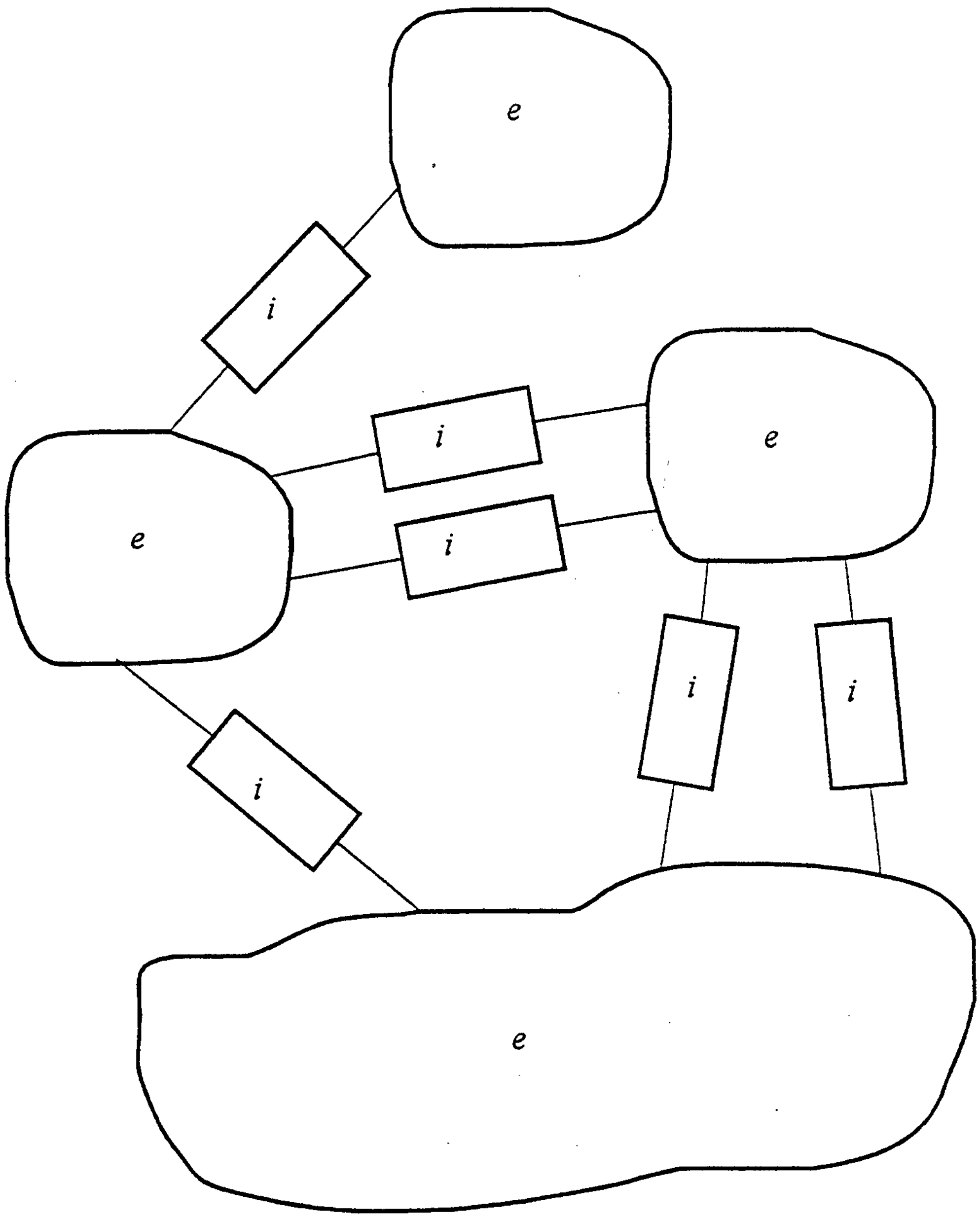


Figure 1-3. General structure of a system with isolators.

Next we give an example of the system with isolators described by a partial differential equation.

1.1.2.1 A System with Isolators Modeled with a Partial Differential Equation. Consider a simplified model of a vehicle that consists of a body and a chassis connected to the body by an isolator (Fig. 1.4). The isolator includes an elastic leaf spring and a hydraulic damper. The chassis consists of a rigid frame and an axle with wheels, the axle being rigidly connected to the frame. The body is assumed to be rigid and the leaf spring is modeled as an elastic beam attached to the frame by means of hinges. The body and damper are attached to the leaf spring at its middle point. The mass of the damper is assumed to be much less than the mass of the other parts of the vehicle, and hence, the inertial properties of the damper are neglected. The damper is linear in that the force generated by the damper and applied to the body is proportional to the velocity of the body relative to the chassis. Vertical oscillations of the leaf spring and the body of the vehicle moving along an uneven road are described by the following differential equations with boundary conditions:

$$\dot{x} = v(t), \tag{1.27}$$

$$\rho\left[\frac{\partial^2 w}{\partial t^2} + \ddot{y}(x(t)) + g\right] + EI\frac{\partial^4 w}{\partial z^4} = 0, \tag{1.28}$$

$$w(0,t) = w(2l,t) = 0, \tag{1.29}$$

$$\frac{\partial^2 w(0,t)}{\partial z^2} = \frac{\partial^2 w(2l,t)}{\partial z^2} = 0, \tag{1.30}$$

$$m\left[\frac{\partial^2 w(l,t)}{\partial t^2} + \ddot{y}(x(t)) + g\right] + c\frac{\partial w(l,t)}{\partial t} = \tag{1.31}$$

$$-EI\left[\frac{\partial^3 w(l+0,t)}{\partial z^3} - \frac{\partial^3 w(l-0,t)}{\partial z^3}\right].$$

Here, x and y are the horizontal and vertical coordinates, respectively, of the point of contact between the wheel and the road; $y(x)$ is a specified function describing the road profile; $v(t)$ is the horizontal component of the vehicle velocity; $2l$ is the length of the leaf spring (beam); $w(z,t)$ is the deflection of the beam; z is the abscissa of a point of the beam neutral axis measured from one of its ends; ρ is the mass per unit length of the beam; E and I are the beam elastic modulus and the cross-section moment of inertia; g is the acceleration due to gravity; m is the mass of the body; and c is the damping factor. By $\ddot{y}(x(t))$ we denote the total second derivative of the function $y(x)$ with respect to time

$$\ddot{y}(x(t)) = v^2(t)\frac{d^2y(x(t))}{dx^2} + \frac{dv(t)}{dt}\frac{dy(x(t))}{dx}, \tag{1.32}$$

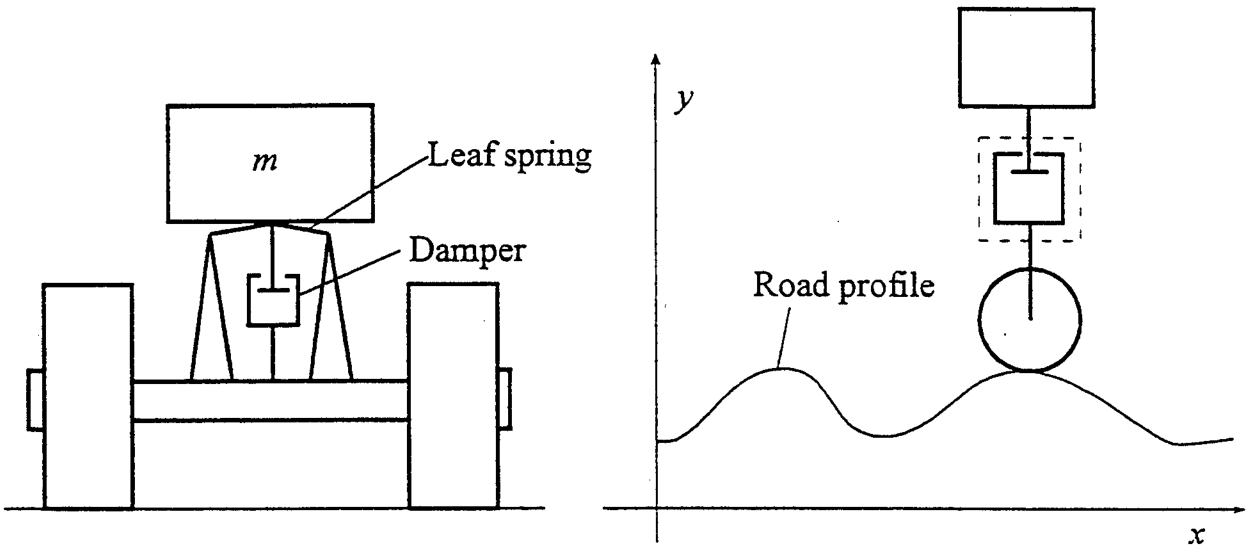


Figure 1-4. A model of a vehicle on an uneven road.

consistent with the variation of the horizontal coordinate x of the vehicle.

Equation (1.27) is an ordinary differential equation describing the horizontal motion of the vehicle. Partial differential equation (1.28) describes bending vibrations of the beam. Relations (1.29) to (1.31) are the boundary conditions for partial differential equation (1.28). Conditions (1.29) and (1.30) are usual deflection and moment boundary conditions, respectively, for a beam hinged at its both endpoints. Condition (1.31) is a shear force relation. This condition defines the jump of the shear force at the beam midpoint to which the vehicle body and the damper are attached. The jump is equal to the concentrated force applied by the beam to the body. Physically, boundary condition (1.31) is Newton's equation of motion for the body under the action of the spring force, specified by the right-hand side in (1.31), the damping force $-c\partial w(l,t)/\partial t$, and the gravity force $-mg$.

To calculate the motion of the system, the differential equations with boundary conditions should be supplemented with initial conditions including the initial shape of the leaf spring, the initial distribution of velocities along the beam axis, and the initial value of the coordinate x.

1.1.2.2 A System of Rigid Bodies with Isolators. Let the mathematical model of the system with isolators contain $N+1$ elements (the base and N objects being isolated). The elements are rigid bodies and the isolators are massless. Systems of this kind are considered in subsequent chapters of this book.

Define the inertial (fixed) reference frame $OXYZ$ and the reference frames $O_iX_iY_iZ_i$ $(i = 0, 1, \dots, N)$ associated with the $N+1$ members of the system. Index $i = 0$ is related to the base, while $i = 1, \dots, N$ correspond to the objects being isolated. To reduce the number of parameters of the mathematical model and to simplify the differential equations of motion, it is advisable to choose the reference frames $O_iX_iY_iZ_i$ so that their coordinate axes coincide with the principal central axes of inertia for corresponding rigid bodies.

Introduce the notation of Fig. 1.5: $\mathbf{R}_0$ is the position vector of the mass center of the base with respect to the inertial reference frame; $\mathbf{r}_{0j}$ is the vector from the mass center of the base to the mass center of the jth object being isolated $(j = 1, \dots, N)$; $L(i,j)$ is the number of isolators connecting the ith and jth members $(i, j = 0, 1, \dots, N)$; $\mathbf{r}_{ij}^k$ is the vector from the mass center of the ith element to a point that belongs to the same body and at which the kth isolator connecting the ith and jth members is attached $(i, j = 0, 1, \dots N;\ i \neq j;\ k = 1, \dots, L(i,j))$; $\mathbf{x}_{ij}^k$ is the vector from the attachment point of the kth isolator connecting the ith and jth elements on the ith member to the attachment point of this isolator on the jth member $(i, j = 0, 1, \dots N;\ i \neq j;\ k = 1, \dots, L(i,j))$; $\mathbf{A}$ is the matrix of transformation from the inertial reference frame $OXYZ$ to the reference frame $O_0X_0Y_0Z_0$ connected with the base; $\mathbf{A}_{ij}$ is the matrix of transformation from the $O_iX_iY_iZ_i$ reference frame to the $O_jX_jY_jZ_j$ reference frame $(i, j = 0, 1, \dots, N)$; $\boldsymbol{\omega}_j$ is the angular velocity of rotation of the jth member of the system relative to the inertial reference frame; m_j is the mass of the jth element $(j = 0, 1, \dots, N)$; J_j is the inertia tensor of the jth element with respect to its mass center $(j = 0, 1, \dots, N)$; $\mathbf{F}_{ij}^k$ is the force applied to the jth member by the kth isolator connecting the ith and jth members $(i, j = 0, 1, \dots, N;\ i \neq j;\ k = 1, \dots, L(i,j))$; $\mathbf{F}_i$ is the net external (i.e. generated by bodies other than members of the system in question) force acting on the ith element of the system with isolators $(i = 0, 1, \dots N)$; and $\mathbf{M}_i$ is the net external moment applied to the ith member, with respect to its mass center

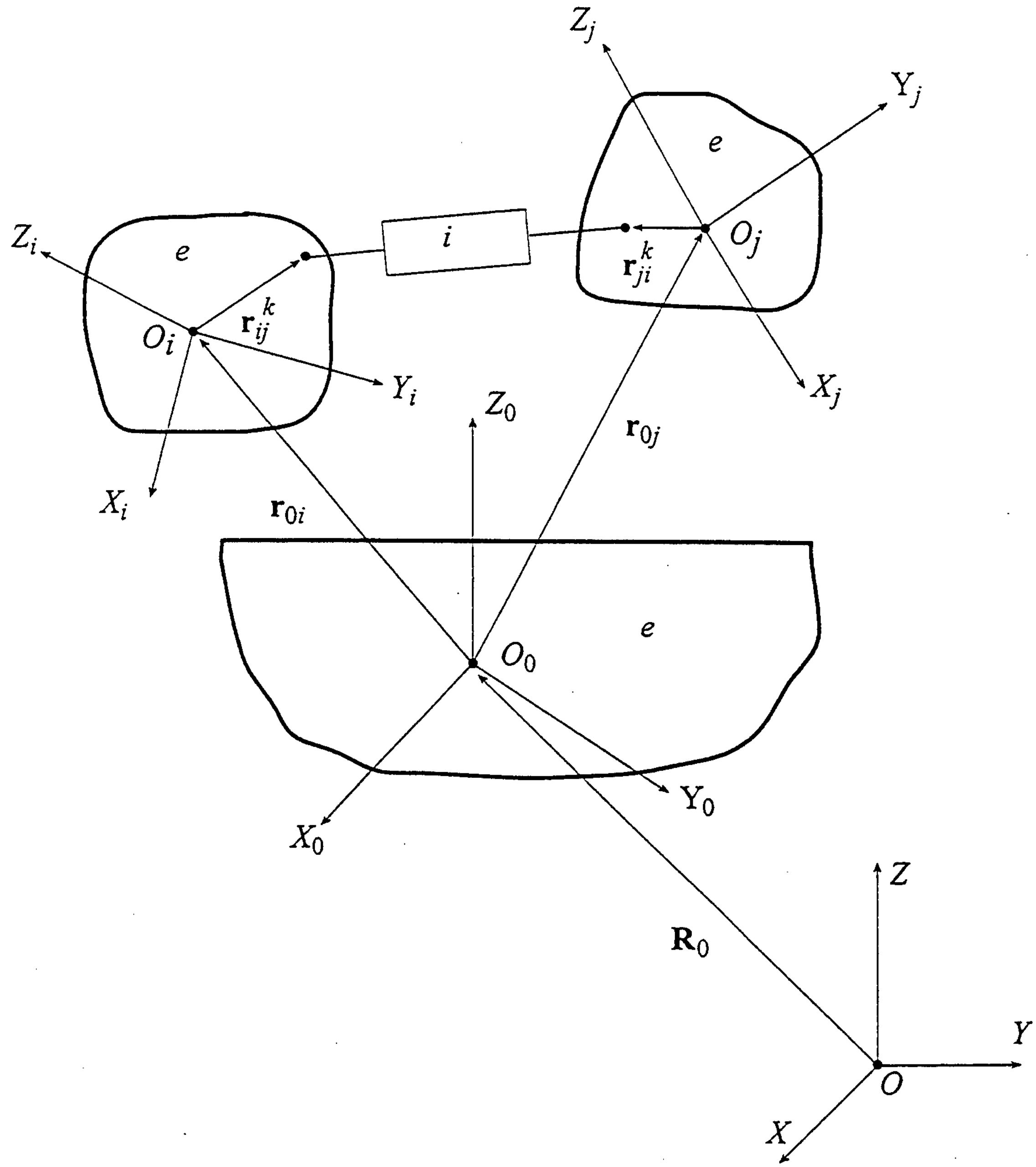


Figure 1-5. Inertial reference frame and the reference frames associated with the elements of a system with isolators. Basic notation.

$(i = 0, 1, \ldots, N)$.

The vectors $\mathbf{r}_{0j}, \mathbf{r}_{ji}^k$, and $\mathbf{x}_{ij}^k$ are related by $\mathbf{x}_{ij}^k = \mathbf{r}_{0j} - \mathbf{r}_{0i} + \mathbf{r}_{ji}^k - \mathbf{r}_{ij}^k$ (see Fig. 1.5), while the transformation matrices $\mathbf{A}_{0i}, \mathbf{A}_{0j}$, and $\mathbf{A}_{ij}$ satisfy the relationship $\mathbf{A}_{ij} = \mathbf{A}_{0j}\mathbf{A}_{0i}^T$. The transpose of a matrix is indicated by superscript T. The set of differential equations governing the motion of the system under consideration in vector-matrix notation is

$$m_0\left[\ddot{\mathbf{R}}_0 + \dot{\boldsymbol{\omega}}_0 \times \mathbf{R}_0 + 2\boldsymbol{\omega}_0 \times \dot{\mathbf{R}}_0 + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \mathbf{R}_0\right] = \sum_{i=1}^{N}\sum_{k=1}^{L(i,0)} \mathbf{F}_{i0}^k + \mathbf{F}_0, \tag{1.33}$$

$$m_j\left[\ddot{\mathbf{r}}_{0j} + \dot{\boldsymbol{\omega}}_0 \times \mathbf{r}_{0j} + 2\boldsymbol{\omega}_0 \times \dot{\mathbf{r}}_{0j} + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_0 \times \mathbf{r}_{0j}\right] = \sum_{i\neq j}\sum_{k=1}^{L(i,j)} \mathbf{F}_{ij}^k - \tag{1.34}$$

$$\frac{m_i}{m_0}\sum_{i=1}^{N}\sum_{k=1}^{L(i,0)} \mathbf{F}_{i0}^k + \mathbf{F}_j - \frac{m_j}{m_0}\mathbf{F}_0,$$

$$\mathbf{J}_l\dot{\boldsymbol{\omega}}_l + \boldsymbol{\omega}_l \times \mathbf{J}_l\boldsymbol{\omega}_l = \sum_{i\neq l}\sum_{k=1}^{L(i,l)} \mathbf{r}_{li}^k \times \mathbf{F}_{il}^k + \mathbf{M}_l, \tag{1.35}$$

$$\dot{\mathbf{A}} = -\boldsymbol{\Omega}_0\mathbf{A}, \quad \dot{\mathbf{A}}_{0j} = \mathbf{A}_{0j}\boldsymbol{\Omega}_0 - \boldsymbol{\Omega}_j\mathbf{A}_{0j}, \quad i,j = 1, ..., N, \tag{1.36}$$

$$\boldsymbol{\Omega}_l = \begin{bmatrix} 0 & -r_l & q_l \\ r_l & 0 & -p_l \\ -q_l & p_l & 0 \end{bmatrix}, \quad l = 0, 1, ..., N$$

Here, p_l, q_l, r_l $(l = 0, 1, \ldots, N)$ are the components of the angular velocity vector $\boldsymbol{\omega}_l$ of the lth element in the $O_lX_lY_lZ_l$ reference frame associated with this element. All vectors in Eqs. (1.33) and (1.34) should be represented by their components in the $O_0X_0Y_0Z_0$ reference frame connected with the base, while in Eq. (1.35) all vectors and the inertia tensor $\mathbf{J}_l$ should be represented in the $O_lX_lY_lZ_l$ reference frame associated with the lth member of the system. Dots above letters denote time derivatives with respect to the reference frame connected with the corresponding body. The transformation of an arbitrary vector $\mathbf{a}$ from the $O_iX_iY_iZ_i$ reference frame to the $O_jX_jY_jZ_j$ reference frame is given by

$$\{\mathbf{a}\}_j = \mathbf{A}_{ij}\{\mathbf{a}\}_i, \quad \mathbf{A}_{ij} = \mathbf{A}_{0j}\mathbf{A}_{0i}^T \tag{1.37}$$

where $\{\mathbf{a}\}_i$ denotes the column of components of the vector $\mathbf{a}$ in the $O_iX_iY_iZ_i$ reference frame. Equations (1.33) and (1.34) govern the motion of the mass centers of the base and the objects being isolated, respectively, whereas Eq. (1.35) describes the change of the angular momenta for

these bodies. The kinematic relations of Eq. (1.36) describe the change of orientation of the base with respect to the inertial reference frame and of the jth object being isolated with respect to the base. The kinematic relationships of Eq. (1.36) expressed in terms of transformation matrices are called Poisson's kinematic equations (Wittenburg, 1977). This is not the only form possible for kinematic equations. The form will change if one chooses for the description of the relative orientation of the system's bodies other variables than the transformation matrix elements, for instance, Euler angles or Rodrigues–Hamilton variables.

Some observations concerning the forces and moments included in the equations of motion are in order. The force $\mathbf{F}_{ij}^k$ is a control force generated by the kth isolator connecting the ith and jth members of the system with isolators. The specific form of the dependence of this force on the variables describing the motion is determined by the structure of the isolator. In the majority of cases encountered in practice, this force depends on the distance between the attachment points of the isolator (on the deformation of the isolation element), the rate of the change of this distance, and (for active isolation systems) on time, i.e. $\mathbf{F}_{ij}^k = \mathbf{F}_{ij}^k(x_{ij}^k, \dot{x}_{ij}^k, t)$. We will call the function $\mathbf{F}_{ij}^k(x_{ij}^k, \dot{x}_{ij}^k, t)$ the *characteristic of the isolator.*

Forces $\mathbf{F}_l$ and moments $\mathbf{M}_l$ ($l = 0, 1, ..., N$) are caused by the action of external bodies (that do not belong to the system with isolators) on the base and objects being isolated. These forces and moments are determined by operating conditions of the system and are not under control of the system's designer and/or operator. In mathematical models of systems with isolators, these forces and moments are often presented as explicit functions of time.

1.1.2.3 General Form of Equations of Motion. Denote by $\mathbf{x}$ the vector of state variables of the system of Eqs. (1.33) to (1.36), by $\mathbf{u}(\mathbf{x}, t)$ the vector consisting of the components of the $\mathbf{F}_{ij}^k$ vectors, and by $\mathbf{v}(t)$ the external disturbance vector containing components of all of the $\mathbf{F}_l$ and $\mathbf{M}_l$ vectors. Then the equations of motion can be represented as the system of the first-order differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}, t), \tag{1.38}$$

Equations of the form of Eq. (1.38) describe the behavior of controlled dynamic systems with a finite number of degrees of freedom in the presence of disturbances. Systems of such a kind are considered in the theory of automatic control (Aizerman, 1966; Roitenberg, 1978), theory of optimal control (Pontryagin, et al., 1962; Lee and Markus, 1967; Boltyanskii, 1968; Krasovskii, 1968; Bryson and Ho, 1975), and theory of differential games (Isaacs, 1965; Krasovskii, 1970; Chernousko and Melikyan, 1978). Methods and results of these mathematical disciplines are applicable to the analysis of systems with isolators and optimal design of isolators. In Kolovskii (1976), systems with isolators are studied from the viewpoint of automatic control. The scope of this book is the analysis of systems with isolators as objects of optimization.

1.1.3 Performance Criteria of Isolation.

The most important mechanical characteristics of systems with isolators are relative displacements of their members, absolute (i.e. related to the inertial reference frame) accelerations of certain locations of the bodies, and forces applied by isolators to their attachment points. The relative displacements of the elements influence the overall dimensions of the structure and its

configuration. The absolute accelerations characterize the forces transmitted to the objects being isolated which can deteriorate normal functioning of the system. The forces generated by isolators at their attachment points influence strength characteristics necessary for the junctions. Functionals of these mechanical quantities can be taken as performance criteria of isolation. The specific forms of these functionals depend on the intended use of a system and its operating conditions.

Several common performance criteria for the design of isolation systems can be identified. Consider a rigid body (an object to be isolated) connected to a movable base by means of M massless isolators. Assume the base simply translates. From Eqs. (1.33) to (1.36) the motion of the body is governed by

$$m_1 \ddot{\mathbf{r}}_{01} = \sum_{k=1}^{M} \mathbf{F}_{01}^k(x_{01}^k, \dot{x}_{01}^k, t) - m_1 \ddot{\mathbf{R}}_0(t), \tag{1.39}$$

$$\mathbf{J}_1 \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \mathbf{J}_1 \boldsymbol{\omega}_1 = \sum_{k=1}^{M} \mathbf{r}_{10}^k \times \mathbf{A}_{01}^k \mathbf{F}_{01}^k, \quad \dot{\mathbf{A}}_{01} = -\boldsymbol{\Omega}_1 \mathbf{A}_{01}$$

$$\mathbf{x}_{01}^k = \mathbf{r}_{01} + \mathbf{A}^T \mathbf{r}_{10}^k - \mathbf{r}_{01}, \quad x_{01}^k = |\mathbf{x}_{01}^k|.$$

The function $\ddot{\mathbf{R}}_0(t)$ describes the acceleration of the base and characterizes the excitation of the system.

Vectors $\mathbf{R}_0, \mathbf{r}_{01}, \mathbf{r}_{01}^k$, and $\mathbf{F}_{01}^k$ in Eq. (1.39) are assumed to be represented in the reference frame associated with the base, whereas vectors $\boldsymbol{\omega}_1$ and $\mathbf{r}_{10}^k$ are represented in the reference frame connected with the body being isolated. It would appear reasonable to represent vectors $\mathbf{r}_{01}^k$ and $\mathbf{r}_{10}^k$ in the reference frames associated with the base and the body being isolated, respectively, since $\mathbf{r}_{01}^k$ and $\mathbf{r}_{10}^k$ are position vectors of the isolator attachment points on the indicated bodies. The initial conditions at $t = t_0$, necessary for solving the system of Eq. (1.39), are

$$\mathbf{r}_{01}(t_0) = \mathbf{r}_{01}^{(0)}, \quad \dot{\mathbf{r}}_{01}(t_0) = \dot{\mathbf{r}}_{01}^{(0)}, \quad \boldsymbol{\omega}_1(t_0) = \boldsymbol{\omega}_1^{(0)}, \quad \mathbf{A}_{01}(t_0) = \mathbf{A}_{01}^{(0)}. \tag{1.40}$$

Next, we wish to express the mechanical characteristics of motion of the body being isolated. Denote by $\boldsymbol{\rho}_z$ the vector from the mass center of the body being isolated to an arbitrary point z of the body. Assume this vector is represented in the $O_1X_1Y_1Z_1$ reference frame associated with the object being isolated.

The position vector $\mathbf{R}_z$ of the point z in the $O_0X_0Y_0Z_0$ reference frame connected with the base and the absolute acceleration $\mathbf{w}_z$ of this point in the $O_1X_1Y_1Z_1$ reference frame are represented as

$$\mathbf{R}_z = \mathbf{r}_{01} + \mathbf{A}_{01}^T \boldsymbol{\rho}_z, \quad \mathbf{w}_z = \mathbf{A}_{0i} \left[\ddot{\mathbf{R}}_0 + \ddot{\mathbf{r}}_{01} \right] + \dot{\boldsymbol{\omega}}_1 \times \boldsymbol{\rho}_z + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_1 \times \boldsymbol{\rho}_z. \tag{1.41}$$

The forces generated by the isolators at their attachment points are determined by the isolator characteristics $\mathbf{F}_{01}^k(x_{01}^k, \dot{x}_{01}^k, t)$.

Denote by G a non-empty set of locations on the body being isolated. Next, we enumerate some functionals that are sometimes used as performance criteria in judging the quality of isolation.

(i) Maximum displacement of points of the body being isolated with respect to the base:

$$J_1 = \max_{z \in G} \max_t |\mathbf{R}_z(t)|. \tag{1.42}$$

(ii) Maximum absolute value of the angle φ of rotation of the body being isolated about an axis fixed on the base, (if such an axis exists):

$$J_2 = \max_t |\varphi(t)|. \tag{1.43}$$

(iii) Maximum magnitude of the absolute acceleration (force) at locations in the body being isolated:

$$J_3 = \max_{z \in G} \max_t |\mathbf{w}_z(t)|. \tag{1.44}$$

(iv) Maximum mean square of the absolute acceleration at the points of the body being isolated:

$$J_4 = \max_{z \in G} \frac{1}{T} \int_0^T |\mathbf{w}_z(t)|^2 \, dt. \tag{1.45}$$

Here, T is a characteristic time of the system. This time is selected by the designer and depends on intended use of the system and its operating conditions.

(v) Maximum absolute value of the force developed by the isolator at the point of attachment to the base or to the body:

$$J_5 = \max_t \left| \mathbf{F}_{01}^k(x_{01}^k(t), \dot{x}_{01}^k(t), t) \right|. \tag{1.46}$$

(vi) Mean square of the absolute value of the force developed by the isolator at the location of its attachment to the base or to the body:

$$J_6 = \frac{1}{T} \int_0^T \left| \mathbf{F}_{01}^k(x_{01}^k(t), \dot{x}_{01}^k(t), t) \right|^2 dt. \tag{1.47}$$

Functionals J_1 and J_2 are geometrical performance criteria. The choice of this functional in the form of the maximum of a geometrical quantity meets practical requirements in that the designer can arrange the system so as to avoid collisions of the body being isolated with parts of the base.

Functionals J_3 to J_6 are dynamical performance criteria, characterizing the loads acting on the base and the body. If the external disturbance is brief, e.g., a shock-like disturbance, then functionals J_3 or J_5 which express a maximum of the corresponding quantity with respect to time are most appropriate. In the case where the external disturbance duration is long, for example, when the base undergoes steady-state vibrations, one can use, along with functionals J_3 and J_5, functionals J_4 and J_6 expressing mean square values of functions characterizing the loading.

Functionals $J_i (i = 1, \ldots, 6)$ are defined on the set of solutions to the equation of motion of Eq. (1.39). For given initial conditions, the performance criteria are functionals of isolator characteristics $\mathbf{F}_{01}^k(x_{01}^k, \dot{x}_{01}^k, t)$ $(k = 1, \ldots, M)$ as well as the external disturbances, i.e., the acceleration $\ddot{\mathbf{R}}_0(t)$ of the base. Modifications of the performance criteria are possible. For example, if the initial value of the state vector of the system of Eq. (1.39) is not specified precisely, but is known to belong to a certain domain G_1, then an additional maximum over all initial values of the state vector from the domain G_1 needs to be involved.

In the general case, where the motion of the system with isolators is governed by Eq. (1.38), mechanical properties of the motion of the object being isolated are usually represented as functions of state variables $\mathbf{x}$, control variables $\mathbf{u}$, the external disturbances $\mathbf{v}$, time t, and the vector $\boldsymbol{\lambda}$ of parameters specifying, for example, the initial conditions of the system motion or identifying a particular location in the body being isolated: $\Phi_i = \Phi_i(\mathbf{x}, \mathbf{u}, \mathbf{v}, t, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \Lambda$. Here, by Λ, the domain of admissible values of the vector $\boldsymbol{\lambda}$ is denoted; i is the number of the characteristic, corresponding to the index numbering of the functionals J_i.

The functionals used for evaluating the quality of isolation when designing and analyzing isolation systems in most cases can be represented by

$$J_i(\mathbf{u}, \mathbf{v}) = \max_{\boldsymbol{\lambda} \in \Lambda} \max_t \Phi_i(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), t), \mathbf{v}(t), t, \boldsymbol{\lambda}) \tag{1.48}$$

or by

$$J_i(\mathbf{u}, \mathbf{v}) = \max_{\boldsymbol{\lambda} \in \Lambda} \int_0^T \Phi_i(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), t), \mathbf{v}(t), t, \boldsymbol{\lambda}) dt. \tag{1.49}$$

Performance criteria of Eqs. (1.48) and (1.49) are functionals of the vector $\mathbf{u}(\mathbf{x}, t)$ of control variables (characteristics of isolators) and the external disturbance vector $\mathbf{v}(t)$.

The performance criteria of Eqs. (1.42) - (1.47) are particular cases of either Eq. (1.48) or Eq. (1.49). Specifically, the criteria J_1, J_2, J_3, and J_5 have the form of Eq. (1.48), whereas the criteria J_4 and J_6 have the form of Eq. (1.49). The functions Φ_i are identified as $\Phi_1 = |\mathbf{R}_z(t)|$, $\Phi_2 = |\varphi(t)|$, $\Phi_3 = |\mathbf{w}_z(t)|$, $\Phi_4 = |\mathbf{w}_z(t)|^2 / T$, $\Phi_5 = \left|\mathbf{F}_{01}^k(x_{01}^k(t), \dot{x}_{01}^k(t), t)\right|$, and $\Phi_6 = \left|\mathbf{F}_{01}^k(x_{01}^k(t), \dot{x}_{01}^k(t), t)\right|^2 / T$. The parameter $\boldsymbol{\lambda}$ in Eqs. (1.42), (1.44), and (1.45) is identified with the position vector of the location z in the body to be isolated and the set Λ is the set of the position

vectors of all $z \in G$. In the criteria of Eqs. (1.43), (1.46), and (1.47), the maximization with respect to the parameter Λ is not performed.

The performance criteria presented here are applicable to deterministic systems whose equations of motion include no stochastic variables. In some cases, the motion of the system with isolators is described by stochastic differential equations. One such case occurs when the external disturbance of the base is regarded as a stochastic process. Performance criteria for such systems are usually expressed as probabilistic functions of corresponding mechanical quantities. For instance, one criterion could be based on the mathematical expectation of the time average of the square of the absolute acceleration at some location in the body. In this book we deal only with deterministic systems.

Monographs by Furunzhiev (1971, 1977) and Larin (1974) and a number of papers (e.g., Bolotin, 1969, 1970; Karnopp and Trikha, 1969) are devoted to the analysis and design of systems with isolators subject to stochastic disturbances.

The functionals discussed in this section quantitatively characterize the quality of the isolation system. The goal in designing the system is to minimize a performance criterion. In what follows, we refer to the performance criterion to be minimized as the *optimization criterion, performance index* or *objective function.* Characteristics and structural parameters of isolators are the design variables.

1.1.4 Active and Passive Isolators.

In the literature on the theory and design of isolation systems, the terms *active* and *passive* occur frequently. These terms sometimes are used in different senses by different authors. In Kolovskii (1976) the distinction is made by whether the isolation system is used for protecting the body being isolated (active system) or the base (passive system) from unfavorable mechanical disturbances. Often, the isolation system is called active if it contains automatic regulators and requires external power supplies, and passive if the isolation system consists only of inertial, elastic, and dissipative elements (Kolovskii, 1976). Thus, such a classification depends on the selection of a particular isolation hardware.

Another criterion for classifying isolation systems into passive and active ones has been suggested by Sevin and Pilkey (1971). They call the isolator active if it depends on time explicitly, and passive otherwise. This criterion is the most formal and it allows easy identification. Moreover, this criterion is convenient, since it does not associate the terms active and passive with a concrete engineering implementation of a particular characteristic, but reflects only the difference in the form of dependence of the force generated by the isolator on the dynamic variables. In this book, the terms active and passive have only this final definition.

1.1.5 External Disturbances.

The external disturbance is the imposition of a force or of prescribed modes of motion on the base or on the objects being isolated by sources that do not belong to the system with isolators. The nature of the external disturbances is determined by operating conditions of this system.

External disturbances can be classified as being *kinematical* or *dynamical.* The external

disturbance is said to be kinematical if it is described by time histories of displacements, velocities, or accelerations of some points of the system. From the viewpoint of analytical dynamics, a kinematical external disturbance is a time-dependent constraint (Gantmakher, 1960) imposed on the bodies of the mechanical system in question. Such a constraint reduces the number of the system's degrees of freedom. If the external disturbance is described by forces applied to the base and/or objects being isolated it is called dynamical.

The most important types of external disturbances considered in the theory of isolation systems are *shock* and *vibration*. In mechanics, shock loading is usually understood to be a short-duration action of a force of considerable magnitude, constant in direction, and having a finite impulse

$$\mathbf{S} = \int_{t_0}^{t_0+\tau} \mathbf{F}(t)dt, \tag{1.50}$$

where $\mathbf{F}(t)$ is the vector function expressing the time history of the force, t_0 and τ are the initial instant and duration of the shock loading. Also, it is assumed that $\mathbf{F}(t) \neq 0$ for $t \in (t_0, t_0+\tau)$ and $\mathbf{F}(t) \equiv 0$ for $t \notin (t_0, t_0+\tau)$. In the case of a kinematical disturbance, shock is understood to be a short-duration acceleration of a point of the base or body being isolated. It has a large absolute value, is constant in direction, and produces a finite increment of the velocity of the point under consideration

$$\Delta\mathbf{v} = \int_{t_0}^{t_0+\tau} \mathbf{w}(t)dt, \tag{1.51}$$

where $\mathbf{w}(t)$ is the acceleration of the point of application of the kinematical disturbance, t_0 and τ are the initial instant and duration of the shock, respectively. Also, $\mathbf{w}(t) \neq 0$ for $t \in (t_0, t_0+\tau)$ and $\mathbf{w}(t) \equiv 0$ for $t \notin (t_0, t_0+\tau)$. The force impulse $\mathbf{S}$ and the velocity increment $\Delta\mathbf{v}$ are basic mechanical characteristics of the shock disturbance.

An important idealization, widely used for mathematical descriptions of shock-like disturbances, is the instantaneous shock (instantaneous impact) at which a finite force impulse (the finite velocity increment, in the case of a kinematical disturbance) is produced for infinitesimal impact duration ($\tau \to 0$). For this case, the absolute value of the force (acceleration, for the kinematical disturbance) at $t = t_0$ tends to infinity so that the values of the integrals in Eqs. (1.50) and (1.51) remain unchanged. Then the force (acceleration, for the kinematical disturbance) can be mathematically described using Dirac's delta function

$$\mathbf{F}(t) = \mathbf{S}\delta(t-t_0) \quad \text{or} \quad \mathbf{w}(t) = \Delta\mathbf{v}\delta(t-t_0). \tag{1.52}$$

The vector coefficient of the delta function is equal to the force impulse (the velocity increment, for the kinematical disturbance) over the shock duration.

The mechanical effect of the instantaneous shock (instantaneous impact) would be a step change in the linear and/or angular momenta of the mechanical system under consideration at the time instant $t = t_0$. For an in-depth discussion of the theory of mechanical shock see, for example, Panovko (1977) and Pars (1979).

In practice, the shock disturbance can be regarded as instantaneous if the duration of the impact is much less than the characteristic time of the system, e.g., the period of the natural vibration of the body being isolated relative to the base, and if the displacements of the system's members for the time τ are much less than characteristic scales (linear and angular) appearing in the problem, e.g., the maximum admissible displacement of the body being isolated relative to the base.

In engineering, the term shock is often understood in a broader sense. In this case, it is not assumed that the shock force is constant in direction, nor is it necessarily equal to zero outside a certain time interval. Any transient disturbance the intensity of which decays relatively rapidly in time is referred to as a shock. For details, see Harris and Crede (1996). A special class of shock disturbances of this sort will be considered in Chapter 4.

Vibration is commonly understood to be an external disturbance with periodic (or almost periodic) variation of the force (acceleration of the kinematical disturbance application point), both in magnitude and direction. Basic mechanical characteristics of vibrations are the amplitude (maximum absolute value) of the force (acceleration of the kinematical disturbance application point) and its frequency. The simplest, and at the same time rather important type of vibrational disturbance, is the harmonic vibration at which the force (acceleration of the kinematical disturbance application point) is directed along some straight line and its magnitude changes in a sinusoidal manner

$$\mathbf{F}(t) = \mathbf{F}_0 \sin(\omega t + \varphi) \quad \text{or} \quad \mathbf{w}(t) = \mathbf{w}_0 \sin(\omega t + \varphi). \tag{1.53}$$

Here $\mathbf{F}_0$ and $\mathbf{w}_0$ are constant amplitude vectors, ω is the vibration frequency, and φ is the initial (at $t = 0$) phase of oscillations.

The harmonic model of vibrational external disturbances is widely used for theoretical analyses of responses of isolation systems and the optimization of their characteristics, see, e.g., Sevin and Pilkey (1971), Wang and Pilkey (1975), and Kolovskii (1976).

1.2 BASIC PROBLEMS OF OPTIMAL ISOLATION.

1.2.1 General Formulation of the Optimization Problem.

Let the motion of the system with isolators in the time interval $t_0 \le t \le T$ be described by the system of differential equations of the form of Eq. (1.38)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t), \mathbf{v}(t), t), \quad t_0 \le t \le T. \tag{1.54}$$

Here, $\mathbf{x}$ is an n-vector of state variables of the system with isolators; $\mathbf{u}(\mathbf{x}, t)$ is an m-vector function of the state variables and time describing the control forces generated by the isolation system (the isolator characteristic); $\mathbf{v}(t)$ is an r-vector function of time describing the external disturbance; $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}, t)$ is a specified n-vector function determined by the system geometric and mechanical structure; and $t_0 \ge 0$ and $T > t_0$ characterize bounds of the time interval in which the motion of the system is considered. If this interval is not bounded from above, set $T = +\infty$.

Denote by U and V the admissible set of the isolator characteristics $\mathbf{u}(\mathbf{x}, t)$ and the set of possible

external disturbances $\mathbf{v}(t)$, respectively. For practical systems with isolators, the set U is based on available design configurations for implementing characteristics $\mathbf{u}(\mathbf{x}, t)$, while the set V is determined by operating conditions of the system as well as by the degree of a designer's knowledge of the external disturbance. We assume that the properties of the function $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}, t)$ and the sets U and V are such that for any pair $(\mathbf{u}(\mathbf{x}, t), \mathbf{v}(t))$ of functions satisfying the conditions $\mathbf{u} \in U$ and $\mathbf{v} \in V$, there exists a unique solution of Eq. (1.54) in the time interval $t_0 \leq t \leq T$, provided the initial condition is specified as

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{1.55}$$

As has been mentioned in Section 1.1.3, the most widely used and important performance criteria of isolation are specified as functionals given by Eqs. (1.48) and (1.49)

$$J(\mathbf{u}, \mathbf{v}) = \max_{\boldsymbol{\lambda} \in \Lambda} \max_{t \in [t_0, T]} \Phi(\mathbf{x}_{u,v}(t), \mathbf{u}(\mathbf{x}_{u,v}(t), t), \mathbf{v}(t), t, \boldsymbol{\lambda}) \tag{1.56}$$

or

$$J(\mathbf{u}, \mathbf{v}) = \max_{\boldsymbol{\lambda} \in \Lambda} \int_{t_o}^{T} \Phi(\mathbf{x}_{u,v}(t), \mathbf{u}(\mathbf{x}_{u,v}(t), t), \mathbf{v}(t), t, \boldsymbol{\lambda}) dt. \tag{1.57}$$

The quantity $\mathbf{x}_{u,v}(t)$ denotes the solution of the initial-value problem of Eqs. (1.54) and (1.55).

We assume that the maxima on the right-hand sides of Eqs. (1.56) and (1.57) exist. This assumption is made in virtually all engineering literature devoted to the theory of isolation systems and optimization of their characteristics, see, for example, Sevin and Pilkey (1971) and Kolovskii (1976). This is justified by the fact that for almost all real systems, isolator characteristics are continuous functions. Also, as a rule, the function Φ does not depend explicitly on time t and the external disturbance $\mathbf{v}(t)$ and is a continuous in $\mathbf{x}$, $\mathbf{u}$, and $\boldsymbol{\lambda}$, while the time T of the system operation is finite. The set Λ of allowable values of the parameter vector $\boldsymbol{\lambda}$ is, as a rule, closed. In the majority of cases, Λ is either the set of points of the body being isolated at which the reduction of the value of Φ is desired or the set of possible initial values $\mathbf{x}_0$ for the state vector $\mathbf{x}$. These conditions guarantee the existence of the maxima in Eqs. (1.56) and (1.57). Note, however, that in a theoretical treatment of optimal design problems for isolation systems, the motion of the system with isolator is often considered in an infinite time interval, and isolator characteristics of a class wider than the class of continuous functions are allowed. Usually, isolator characteristics are confined to the class of piecewise continuous functions $\mathbf{u}(\mathbf{x}, t)$. For functions $\mathbf{u}(\mathbf{x}, t)$ belonging to this class, the existence of the maximum with respect to time in Eq. (1.56) is not guaranteed *a-priori*. Therefore, for more generality, the operation of calculating maximum ($\max_t$) should have been replaced by the operation of calculating supremum ($\sup_t$). However, to avoid unnecessary complications when presenting the material of this book we retain the notation $\max_t$ in Eq. (1.56). Here, we do not deal with specific features associated with the difference between operations $\max_t$ and $\sup_t$. For the optimization problems considered in subsequent chapters of the book, the maximum of Φ with respect to time exists.

By virtue of the assumed uniqueness of the solution of the initial-value problem of Eqs. (1.54) and (1.55), to each pair of functions $\mathbf{u}(\mathbf{x}, t) \in U$ and $\mathbf{v}(t) \in V$ there corresponds a unique value of J defined by Eqs. (1.56) or (1.57). Thus, relationships of Eqs. (1.56) and (1.57) define functionals in the Cartesian product $U \times V$ of the set of allowable isolator characteristics and the set of possible external disturbances.

Isolation systems intended for the protection of engineering structures from shock and vibration must, as a rule, meet a number of requirements necessary for normal operation of the system. Very often, these requirements are competing. For example, the suspension system of a car must provide a considerable reduction in the average level of the absolute value of acceleration of the car body's mass center. This mitigates the loads acting on parts rigidly connected to the body, as well as on the driver and passengers, increasing thereby the reliability and comfort of the car. At the same time, to provide high controllability and stability of the car, the suspension system must not allow large displacements of the body mass center relative to the road profile, as well as large angular oscillations of the body. Therefore, when designing such systems, one, as a rule, has to take into account several performance criteria of the type of Eqs. (1.56) and/or (1.57).

One approach for the rational calculation of characteristics of isolation systems involves their optimization with respect to a performance criterion. Let N functionals $J_i(\mathbf{u}, \mathbf{v})$, $i = 1, ..N$, of the form of Eqs. (1.56) and/or (1.57) be chosen for evaluating the performance quality of the isolation system. For example, when calculating characteristics of the car suspension system. the functionals $J_i(\mathbf{u}, \mathbf{v})$ can be chosen to be the mean square of the absolute (i.e. related to the inertial reference frame) acceleration of the mass center of the body, the maximum absolute value of the displacement of the body mass center relative to the road surface, and the mean value of the angle of rotation of the body with respect to its mass center. The lower the numerical values of the functionals $J_i(\mathbf{u}, \mathbf{v})$, the better the suspension system, and it is to be expected that the designer will try to reduce these functionals by using springs and dampers. However, in the majority of cases, the functionals $J_i(\mathbf{u}, \mathbf{v})$ reflect contradictory (competing) requirements for the system being designed, and their simultaneous minimization is impossible. An example is the common problem of trying to reduce the maximum absolute value of acceleration with respect to the inertial reference frame for the object being isolated mounted on a movable base and to reduce simultaneously the maximum displacement of the object relative to the base. The most frequently employed approach is to calculate the isolator characteristics by using optimization methods in which one of the N performance criteria, say $J_1(\mathbf{u}, \mathbf{v})$, is to be minimized while the others $(J_i(\mathbf{u}, \mathbf{v}), i = 2, \ldots, N)$ are constrained by specified bounds corresponding to acceptable conditions of operation. In what follows, the performance criterion to be minimized is called the *optimization criterion (performance index, objective function)*. The decision as to which of the functionals $J_i(\mathbf{u}, \mathbf{v}), i = 1, \ldots, N$, should be designated as the optimization criterion depends on which functional appears to be critical to the quality of the system performance and the available information on the admissible upper bounds for the functionals $J_i(\mathbf{u}, \mathbf{v})$, $i = 1, \ldots, N$. For example, suppose the isolation system for a device to be installed in an aircraft is being designed and the maximum transmitted force that this device can tolerate without failure is known. Then the maximum displacement of the device relative to the aircraft body can be chosen the optimization criterion and the corresponding constraint on the transmitted force can be imposed. Often, such a choice of the optimization criterion corresponds to the requirement of there being limited space available for the device in the aircraft. In the overwhelming majority of cases, the

functionals $J_i(\mathbf{u}, \mathbf{v}), i = 1, \ldots, N$, are maxima or integrals of absolute values or squares of responses of the system and, hence, are nonnegative. The conditions of normal operation of systems with isolators are usually expressed as the requirement of not exceeding specified maximum admissible values by some of the functionals $J_i(\mathbf{u}, \mathbf{v})$. Without essential loss of generality, we will consider constraints of this sort only.

Let us proceed to the strict mathematical formulation of problems of optimization of isolation system characteristics in accordance with the approach outlined above. Denote

$$\widetilde{J}_i(\mathbf{u}) = \max_{\mathbf{v} \in V} J_i(\mathbf{u}, \mathbf{v}), \quad i = 1, \ldots, N. \tag{1.58}$$

The quantities $\widetilde{J}_i(\mathbf{u}), i = 1, .., N$, are functionals depending only on the isolator characteristic $\mathbf{u}$, for which the system performance quality provided by the characteristic $\mathbf{u} \in U$ is evaluated under the least favorable of the external disturbances $\mathbf{v} \in V$. Note that in the general case, the least favorable external disturbances (worst disturbances) are different for different $J_i(\mathbf{u}, \mathbf{v})$. If the set V of possible external disturbances consists of a single element $\mathbf{v} = \mathbf{v}_0(t)$, then, obviously, $\widetilde{J}_i(\mathbf{u}) = J_i(\mathbf{u}, \mathbf{v}_0)$.

Sometimes, we will refer to the quantity $\widetilde{J}_i(\mathbf{u})$ of Eq. (1.58) as the *guaranteed value* of the functional $J_i(\mathbf{u}, \mathbf{v})$ for the given $\mathbf{u}$. This term is justified by the fact that none of the external disturbances $\mathbf{v} \in V$ can make the value of J_i greater than $\tilde{J}_i$ and in this sense, the value $\tilde{J}_i$ is ensured (guaranteed) for the specified $\mathbf{u}$.

In a rather general form, the problem of choosing the optimal characteristic for the isolator is formulated as follows.

1.2.1.1 Problem 1.1. From among a specified class U of admissible isolator characteristics, determine the optimal characteristic $\mathbf{u}_0 \in U$ that minimizes the functional $\widetilde{J}_1(\mathbf{u})$, provided the other functionals $\widetilde{J}_i(\mathbf{u}), i = 2, \ldots, N$ do not exceed prescribed maximum allowable values $D_i, i = 2, \ldots, N$, i.e.

$$\widetilde{J}_1(\mathbf{u}_0) = \min_{\mathbf{u} \in U} \{\widetilde{J}_1(\mathbf{u}) | \tilde{J}_i(\mathbf{u}) \leq D_i, \quad i = 2, \ldots, N\}. \tag{1.59}$$

The expression in curly brackets means that the functional $\widetilde{J}_1(\mathbf{u})$ is considered only for those $\mathbf{u} \in U$ that do not violate the constraints $\widetilde{J}_i(\mathbf{u}) \leq D_i, \quad i = 2, \ldots, N$.

It follows from the definition of a *guaranteed value* for the quantity $\tilde{J}_1$ of Eq. (1.58) that it is appropriate to call the quantity $\tilde{J}_1(\mathbf{u}_0)$ of Eq. (1.59) the *guaranteed minimum* of the functional $J_1(\mathbf{u}, \mathbf{v})$.

The statement of Problem 1.1 concerning optimization of the isolator characteristic takes into account the incompleteness of information about the external disturbance. To solve this problem, one does not require the exact knowledge of the external disturbance, which is rarely available in practice. It is sufficient to describe the set of admissible disturbances each one of which can occur when operating the system. The solution of Problem 1.1 yields the optimal characteristic $\mathbf{u}_0 \in U$

ensuring the guaranteed (i.e. not exceedable under the least favorable external disturbance $\mathbf{v} \in V$) minimum of the optimization criterion $J_1(\mathbf{u}, \mathbf{v})$, provided the other performance criteria $J_i(\mathbf{u}, \mathbf{v}), i = 2, \ldots, N$, do not exceed prescribed values $D_i, i = 2, \ldots, N$.

1.2.1.2 Reciprocal Optimization Problems. In the study of the design of isolation systems, the most simple case, and at the same time rather commonly occurring, is when the requirements of the performance quality are characterized by only two functionals, $J_1(\mathbf{u}, \mathbf{v})$ and $J_2(\mathbf{u}, \mathbf{v})$, i.e. $N = 2$. For this case, let us consider two problems of optimization, one of which is based on $J_1(\mathbf{u}, \mathbf{v})$ as the optimization criterion, and for the other, $J_2(\mathbf{u}, \mathbf{v})$ is the optimization criterion.

1.2.1.3 Problem 1.2. Find the optimal characteristic $\mathbf{u}_0^{D_2} \in U$ such that

$$\widetilde{J}_1(\mathbf{u}_0^{D_2}) = \{\widetilde{J}_1(\mathbf{u}) \mid \widetilde{J}_2(\mathbf{u}) \leq D_2\}. \tag{1.60}$$

1.2.1.4 Problem 1.3. Find the optimal characteristic $\mathbf{u}_{D_1}^0 \in U$ such that

$$\widetilde{J}_2(\mathbf{u}_{D_1}^0) = \{\widetilde{J}_2(\mathbf{u}) \mid \widetilde{J}_1(\mathbf{u}) \leq D_1\}. \tag{1.61}$$

The superscript D_2 and subscript D_1 indicate the dependence of the characteristics on the parameters D_1 and D_2 which describe the constraints.

Under certain conditions (which will be discussed later) that are fulfilled in many cases of practical importance, one can utilize the solution of one of Problems 1.2 or 1.3 to obtain the solution of the other. In the literature, such a property of optimization is called *duality* or *reciprocity* (Sevin and Pilkey, 1971).

Introduce the notation

$$g(D_2) = \widetilde{J}_1(\mathbf{u}_0^{D_2}), \quad h(D_1) = \widetilde{J}_2(\mathbf{u}_{D_1}^0). \tag{1.62}$$

where functions $g(D_2)$ and $h(D_1)$ express the dependence of the minimum value of the optimization criterion on the maximum allowable value of the performance criterion subject to the constraint. The reciprocity of Problems 1.2.and 1.3 can be established by

1.2.1.5 Theorem 1.1. Let the one-parameter family of optimal characteristics $\mathbf{u}_0^{D_2}$ (D_2 is the parameter of the family) corresponding to Problem 1.2 be known. Let, in addition, the equality $J_2(\mathbf{u}_0^{D_2}) = D_2$ hold for any admissible value of D_2 and the function $g(D_2)$ of Eq. (1.62) be continuous and monotonically decreasing. Then the optimal characteristic $\mathbf{u}_{D_1}^0$ in Problem 1.3 is determined by $\mathbf{u}_{D_1}^0 = \mathbf{u}_0^{g^{-1}(D_1)}$.

The notation g^{-1} stands for the inverse of the function g. The existence of the inverse follows from the continuity and monotonicity of the function $g(D_2)$.

Proof. Let us assume the opposite, i.e., that the characteristic $\mathbf{u}_{D_1}^0 = \mathbf{u}_0^{g^{-1}(D_1)}$ is not optimal for Problem 1.3. This means that there exists a characteristic $\mathbf{u}_* \in U$ satisfying the inequalities

$$\tilde{J}_1(\mathbf{u}_*) \le D_1, \quad \tilde{J}_2(\mathbf{u}_*) < \tilde{J}_2(\mathbf{u}_0^{g^{-1}(D_1)}). \tag{1.63}$$

According to the condition of the theorem,

$$\tilde{J}_2(\mathbf{u}_0^{g^{-1}(D_1)}) = g^{-1}(D_1). \tag{1.64}$$

Since the function g is monotonically decreasing, Eq. (1.64) and the second inequality in Eq. (1.63) imply the relation

$$g(\tilde{J}_2(\mathbf{u}_*)) > D_1. \tag{1.65}$$

The definition of the function g given by Eq. (1.62) immediately leads to the equalities

$$g(\tilde{J}_2(\mathbf{u}_*)) = \tilde{J}_1(\mathbf{u}_0^{\tilde{J}_2(u_*)}) = \min_{u \in U}\{\tilde{J}_1(\mathbf{u}) | \tilde{J}_2(u) \le \tilde{J}_2(\mathbf{u}_*)\}. \tag{1.66}$$

The latter relationship and Eq. (1.65) imply that

$$\tilde{J}_1(\mathbf{u}_*) \ge g(\tilde{J}_2(\mathbf{u}_*)) > D_1. \tag{1.67}$$

Thus, the assumption that the characteristic $\mathbf{u}_{D_1}^0 = \mathbf{u}_0^{g^{-1}(D_1)}$ is not optimal for Problem 1.3 contradicts the inequality $\tilde{J}_1(\mathbf{u}_*)) \le D_1$. This completes the proof of the theorem.

It follows from Theorem 1.1 that solutions of Problems 1.2 and 1.3 are related by

$$\mathbf{u}_{D_1}^0 = \mathbf{u}_0^{g^{-1}(D_1)}, \quad \tilde{J}_2(\mathbf{u}_{D_1}^0) = h(D_1) = g^{-1}(D_1), \tag{1.68}$$

$$\mathbf{u}_0^{D_2} = \mathbf{u}_{h^{-1}(D_1)}^0, \quad \tilde{J}_1(\mathbf{u}_0^{D_2}) = g(D_2) = h^{-1}(D_2). \tag{1.69}$$

With these relations, given the solution of one of the Problems 1.2 or 1.3, one can easily obtain the solution to the other problem.

The reciprocity property of Problems 1.2 and 1.3 is rather useful when calculating and analyzing optimal characteristics for isolators of different types. It is repeatedly used in subsequent chapters of this book. From a mathematical point of view, problems of determining optimal characteristics of isolators are rather general problems of mathematical programming. The choice of practical methods for solving the problems is determined by (1) specific features of the set of admissible characteristics and the set of possible disturbances, (2) the structure of the system being designed that determines the dimensions of the state-variable and control-variable vectors, (3) the form of the function $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}, t)$, and (4) the form of functionals used as performance criteria. Next, we discuss particular optimal isolation problems and methods of solution.

1.2.2 The Problem of Limiting Isolation Capabilities.

1.2.2.1 General Formulation of the Problem. In this section, we consider the case where the external disturbance $\mathbf{v}(t)$ is known as a function of time defined over the interval $t_0 \leq t \leq T$. Thus, the set V contains only one element. Omit the symbol $\mathbf{v}(t)$ in the list of arguments of the function $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}, t)$. For simplicity, assume that the operation of calculating the maximum with respect to $\boldsymbol{\lambda}$ is absent from the functionals of Eqs. (1.56) and (1.57) so that these functionals are given by

$$J(\mathbf{u}) = \max_{t \in [t_0, T]} \Phi(\mathbf{x}_u(t), \mathbf{u}, t) \tag{1.70}$$

or

$$J(\mathbf{u}) = \int_{t_0}^{T} \Phi(\mathbf{x}_u(t), \mathbf{u}, t) dt \tag{1.71}$$

When designing isolators in practice, their structure is usually subject to restrictions, which are taken into account in the definition of the set U of admissible isolator characteristics. For example, very often it is required to provide the protection of the object from unfavorable external disturbances by using only passive isolators, characteristics of which do not depend on time explicitly. In this case, it is reasonable to take as U the set of all piecewise continuous functions of the state variables. If the structure under design is restricted to the use of only spring-and-damper isolators with elastic and dissipative properties (see Chapter 2), then the set U is a parametric family of functions of the state variables. Parameters of this family are the stiffness and damping coefficients that are to be determined by a designer. However, in all cases, the realization of any isolator characteristic $\mathbf{u}(\mathbf{x}, t) \in U$ for the motion of the system of Eq. (1.54) with the initial conditions of Eq. (1.55) is a function of time $\mathbf{u}(\mathbf{x}_u(t), t) = \bar{\mathbf{u}}(t)$ defined over the interval $t_0 \leq t \leq T$. Therefore, it is of interest to consider the problem of choosing the optimal isolator characteristic from among a rather wide class of functions that depend only on time and are subject to constraints reflecting only the conditions of implementability of the system under design. For engineering practice, it is quite sufficient to seek the optimal characteristic in the class of piecewise continuous functions. Assume that the admissible characteristics are continuous at the boundary points $t = t_0$ and $t = T$ and are continuous on the right at internal points of the interval $[t_0, T]$. The problem of optimal isolation corresponding to the admissible characteristic set U defined in such a way is called the *problem of limiting isolation capabilities* or the *limiting performance problem* because its solution results in the calculation of the absolute minimum of the optimization criterion under specified initial conditions and constraints $J_i(\mathbf{u}) \leq D_i$ on criteria other than the optimization criterion. This minimum cannot in any way be reduced, thus showing the limiting capabilities of isolation for the specified external disturbance, initial conditions, and constraints. See, for example, Guretskii (1965b, 1969b) and Sevin and Pilkey (1971).

Let us present the mathematical formulation of the problem of limiting isolation capabilities.

1.2.2.2 Problem 1.4. Let the motion of the system with isolators be governed by the following vector differential equation and initial conditions:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t_0 \leq t \leq T, \tag{1.72}$$

$$\mathbf{x} = (x_1, \ldots, x_n), \quad \mathbf{u} = (u_1, \ldots, u_m), \quad \mathbf{f} = (f_1, \ldots, f_n).$$

From among the set of piecewise continuous functions of time, find the optimal characteristic $\mathbf{u}_0(t)$ such that

$$J_1(\mathbf{u}_0) = \min_u \{J_1(\mathbf{u}) | J_i(\mathbf{u}) \leq D_i, \quad i = 2, \ldots, N\}. \tag{1.73}$$

The meaning and dimensions of the variables in Eq. (1.72) coincide with those of the corresponding quantities in Eqs. (1.54) and (1.55).

From a mathematical point of view, the problem of limiting isolation capabilities is the problem of determining the open-loop optimal control subject to constraints on the control and state variables. These constraints are indirectly expressed by the inequalities $J_i(\mathbf{u}) \leq D_i$. Methods for solving optimal control problems are presented in many textbooks and monographs devoted to the calculus of variations and the theory of optimal control. Representative books include Pontryagin, et al. (1962); Lee and Markus (1967); Boltyanskii (1968); Young (1969); Bryson and Ho (1975); Leitmann (1981).

In a simpler case, there are no constraints on the state variables, and the constraints on the control function require that the values of the vector function $\mathbf{u}(t)$ at each instant belong to a specified bounded closed domain $\mathbf{U} : \mathbf{u}(t) \in \mathbf{U} \subset E^m$. Here, E^m denotes an m-dimensional Euclidean space. One should distinguish between the domain $\mathbf{U}$ and the set U of admissible characteristics introduced earlier. The set U is defined in a functional space and determines such properties of admissible characteristics as continuity, differentiability, and the presence or absence of explicit dependence on a particular argument. The domain $\mathbf{U}$ is specified in a Euclidean space of dimension m, which coincides with the dimension of the control vector, and characterizes the possibility or impossibility of the assumption by the vector $\mathbf{u}(t)$ of particular values.

1.2.2.3 Simplest Problem of Optimal Isolation. Consider the system shown in Fig. 1.1 which consists of a body being isolated connected with the rectilinearly moving base by means of an isolator. The isolator allows displacement of the body relative to the base in the direction of the latter's motion. The motion of the body with respect to the base is described by the scalar differential equation of Eq. (1.2). Introduce the notation $u = -g(x, \dot{x}, t)/m$, $v = -\ddot{y}$ to represent Eq. (1.2) in the form

$$\ddot{x} + u = v(t). \tag{1.74}$$

In Eq. (1.74), x is the coordinate describing the relative displacements of the object (body) being isolated; $-u$ is the control force (the isolator characteristic) applied to the object being isolated by the isolator, divided by the mass of the object; $-v(t)$ is the acceleration of the base with respect to an inertial reference frame. The quantity $v(t)$ describes the external disturbance.

We assume that Eq. (1.74) is subject to zero initial conditions at the instant $t = 0$

$$x(0) = \dot{x}(0) = 0. \tag{1.75}$$

For the isolation performance criteria, we take the maximum absolute value of the displacement of the body relative to the base,

$$J_1(u) = \max_{t\in[0,T]} |x_u(t)|\,, \tag{1.76}$$

and the maximum absolute value of the force acting upon the body,

$$J_2(u) = \max_{t\in[0,T]} |u(t)|\,. \tag{1.77}$$

Subscript u of the variable x in Eq. (1.76) indicates the dependence of the solution of Eq. (1.74) on the choice of the control. Let the functional of Eq. (1.76) be the optimization criterion. Then Problem 1.4 of limiting isolation capabilities when applied to the system of Eq. (1.74) becomes:

1.2.2.4 Problem 1.5. For the system of Eq. (1.74) with the initial conditions of Eq. (1.75), find a control function $u_0(t)$ which is piecewise continuous, continuous on the right at internal points of the interval $[0,T]$, and such that

$$J_1(u_0) = \max_{t\in[0,T]} |x_{u_0}(t)| = \min_u J_1(u) = \min_u \max_{t\in[0,T]} |x_u(t)|\,, \tag{1.78}$$

provided

$$J_2(u) = \max_{t\in[0,T]} |u(t)| \leq u_*. \tag{1.79}$$

Here, u_* is a specified positive number. Such a problem is considered in Guretskii (1965a) and Sevin and Pilkey (1971) and is sometimes called the *simplest problem of optimal isolation* (Kolovskii, 1976).

Denote state variables (relative displacement, x, and velocity, $\dot{x}$, of the body being isolated) of the single-degree-of-freedom system in question by x_1 and x_2, and then reduce the second-order differential equation of Eq. (1.74) to the set of first-order differential equations of the form of Eq. (1.72), where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{f}(\mathbf{x},\mathbf{u},t) = \begin{bmatrix} x_2 \\ v(t) - u(t) \end{bmatrix}, \quad n = 2, r = 1, m = 1. \tag{1.80}$$

The constraint of Eq. (1.79) is equivalent to the inequality $|\, u(t) \,| \leq u_*$, $t \in [0,T]$, and Problem 1.5 can be represented as the optimal control problem without constraints on the state variables and with constraints on the control given by $u(t) \in \mathbf{U}$, where $\mathbf{U} = [-u_*, u_*]$.

1.2.2.5 Pontryagin's Maximum Principle. Suppose there are no state variable constraints and the control constraints are given by $\mathbf{u}(t) \in \mathbf{U}$. Also, assume the optimization criterion is the integral functional of the form of Eq. (1.71). Then, one can use Pontryagin's maximum principle for solving the problem of limiting isolation capabilities. Pontryagin's maximum principle is one of

the basic theorems of optimal control theory. It expresses necessary optimality conditions. Next, we will give a rather general formulation of the maximum principle. For the proof see Pontryagin, et al. (1962), Boltyanskii (1968), Moiseev (1975), and others.

Suppose that the system behavior is governed by the differential equation with initial conditions of Eq. (1.72) and the functional to be minimized is represented by

$$J(\mathbf{u}) = \int_{t_0}^{T} \Phi_1(\mathbf{x}(t), \mathbf{u}(t), t)dt + \Phi_2(\mathbf{x}(T), T). \tag{1.81}$$

Also, suppose that the set of admissible controls is the set of piecewise continuous functions defined over the interval $[t_0, T]$, there are no constraints on the state variables, and the control constraints are given by $\mathbf{u}(t) \in \mathbf{U}$, where $\mathbf{U} \in E^m$ is a bounded closed set. Moreover, let the state vector $\mathbf{x}(t)$ of the system at the final instant $t = T$ (which is not prescribed in advance in the general case) belong to a terminal manifold S specified by the equations

$$g_i(\mathbf{x}(T), T) = 0, \quad i = 1, \ldots, k, \quad k \leq n, \tag{1.82}$$

where $g_i(\mathbf{x}, t)$ are known functions.

We assume that the functions $f_i(\mathbf{x}, \mathbf{u}, t)$, $i = 1, \ldots, n$, and their partial derivatives $\partial f_i(\mathbf{x}, \mathbf{u}, t)/\partial x_j, (i, j = 1, \ldots, n)$ are continuous and satisfy the Lipschitz condition (Tikhonov, et al., 1980) in the variables $\mathbf{x}$ and $\mathbf{u}$. Also, assume that the functions $\Phi_1(\mathbf{x}, \mathbf{u}, t), \Phi_2(\mathbf{x}, t)$, and $g_l(\mathbf{x}, t)$, $l = 1, \ldots, k$, are continuous and have continuous partial derivatives $\partial\Phi_1/\partial x_j, \partial\Phi_2/\partial x_j, \partial\Phi_2/\partial t, \partial g_l/\partial x_j$, and $\partial g_l/\partial t$ $(j = 1, \ldots, n; l = 1, \ldots, k)$.

Introduce the Hamiltonian function

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}, t, \mathbf{u}) = \mathbf{p}^T\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - p_0\Phi_1(\mathbf{x}, \mathbf{u}, t) = \textstyle\sum_{j=1}^{n} p_j f_j(\mathbf{x}, \mathbf{u}, t) - \\ p_0\Phi_1(\mathbf{x}, \mathbf{u}, t), \end{aligned} \tag{1.83}$$

$$p_0 \geq 0, \quad p_0 = \text{const}.$$

Here, $\mathbf{p} = \mathbf{p}(t) = [p_1(t), \ldots, p_n(t)]^T$ is the vector of adjoint variables satisfying the adjoint system of differential equations

$$\dot{p}_i = -\frac{\partial H(\mathbf{x}_u(t), \mathbf{p}, t, \mathbf{u}(t))}{\partial x_i} = -\sum_{j=1}^{n} p_j \frac{\partial f_j(\mathbf{x}_u(t), \mathbf{u}(t), t)}{\partial x_i} + p_0 \frac{\partial \Phi_1(\mathbf{x}_u(t), \mathbf{u}(t), t)}{\partial x_i}, \tag{1.84}$$

$$i = 1, \ldots, n.$$

To calculate the vector $\mathbf{p}(t)$ of Eq. (1.84) one should first integrate the system of Eq. (1.72) for a given control $\mathbf{u}(t), t \in [t_0, T]$, and then substitute the corresponding solution $\mathbf{x}_u(t)$ and the control $\mathbf{u}(t)$ into the right-hand side of Eq. (1.84). Thus, the vector $\mathbf{p}(t)$ depends on the control $\mathbf{u} = \mathbf{u}(t)$.

Sometimes we will use the superscript u when denoting the adjoint vector ($\mathbf{p}(t) = \mathbf{p}^u(t)$), in order to show this dependence.

1.2.2.6 Theorem 1.2. (Pontryagin's maximum principle). Let $\mathbf{u}_0(t)$ be the optimal control. Then there exist a continuous adjoint vector function $\mathbf{p}^{u_0}(t) = [p_1^{u_0}(t), \ldots, p_n^{u_0}(t)]^T$, a non-negative constant p_0, and constants $\lambda_i, i = 1, \ldots, k$, that satisfy the following conditions:

1) at least one of the quantities $p_i, i = 0, 1, \ldots, n$ is not equal to zero identically over the interval $[t_0, T]$;

2) adjoint variables $\mathbf{p}_i^{u_0}(t), i = 1, \ldots, n$, satisfy Eq. (1.84);

3) the relationship

$$H(\mathbf{x}_{u_0}(t), \mathbf{p}^{u_0}(t), t, \mathbf{u}_0(t)) = \max_{u \in \mathbf{U}} H(\mathbf{x}_{u_0}(t), \mathbf{p}^{u_0}(t), t, \mathbf{u}) \tag{1.85}$$

holds for all $t \in [t_0, T]$;

4) at the final time instant $t = T$, the boundary conditions

$$g_l(\mathbf{x}_{u_0}(T), T) = 0, \quad l = 1, \ldots, k; \tag{1.86}$$

$$p_i^{u_0}(T) = \sum_{j=1}^{k} \lambda_j \frac{\partial g_j(\mathbf{x}_{u_0}(T), T)}{\partial x_i} - p_0 \frac{\partial \Phi_2(\mathbf{x}_{u_0}(T), T)}{\partial x_i}, i = 1, \ldots, n; \tag{1.87}$$

$$\sum_{j=1}^{k} \lambda_j \frac{\partial g_j(\mathbf{x}_{u_0}(T), T)}{\partial t} + H(\mathbf{x}_{u_0}(T), \mathbf{p}^{u_0}(T), T, \mathbf{u}_0(T)) = p_0 \frac{\partial \Phi_2(\mathbf{x}_{u_0}(T), T)}{\partial t} \tag{1.88}$$

are satisfied.

In the literature on the calculus of variations and the theory of optimal control, Eqs. (1.87) and (1.88) are called the transversality conditions. If among the functions $g_l(\mathbf{x}, t), l = 1, \ldots, k$, specifying the terminal manifold, there exists a function, say, $g_1(\mathbf{x}, t)$, that varies monotonically in time along trajectories of the system of Eq. (1.72) and has non-zero time derivatives according to Eq. (1.72), then this function can be used as a condition of the process termination. Then, the final instant $t = T$ is determined as the first root of the equation $g_1(\mathbf{x}_u, t) = 0$. If the function $g_1(\mathbf{x}, t)$ possesses the aforementioned properties, the transversality conditions of Eqs. (1.87) and (1.88) are equivalent to the following relationships:

$$\mathbf{p}_i^{u_0}(T) = \left[p_0 \Phi_1(\mathbf{x}_{u_0}(T), \mathbf{u}_0(T), T) + p_0 \frac{d\Phi_2(\mathbf{x}_{u_0}(T), T)}{dT} - \sum_{j=2}^{k} \lambda_j \frac{dg_j(\mathbf{x}_{u_0}(T), T)}{dT} \right] \times$$

$$\frac{\partial g_1(\mathbf{x}_{u_0}(T),T)}{\partial x_i}\left[\frac{dg_1(\mathbf{x}_{u_0}(T),T)}{dT}\right]^{-1}+\sum_{j=2}^{k}\lambda_j\frac{\partial g_j(\mathbf{x}_{u_0}(T),T)}{\partial x_i}-$$

$$p_0\frac{\partial \Phi_2(\mathbf{x}_{u_0}(T),T)}{\partial x_i}, \quad i=1,\ldots,n. \tag{1.89}$$

Here, d/dt means the operator

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\sum_{i=1}^{n} f_i(\mathbf{x},\mathbf{u},t)\frac{\partial}{\partial x_i} \tag{1.90}$$

of differentiation with respect to time along the trajectories of Eq. (1.72).

Note that the condition of the process termination reduces by one the number of the transversality conditions and the number of constant multipliers λ_i in the formulation of the maximum principle. There are n conditions of Eq. (1.89) instead of $n+1$ relationships of Eqs. (1.87) and (1.88). When deriving Eq. (1.89) from Eqs. (1.87) and (1.88) the constant λ_1 is expressed in terms of other variables.

If $p_0 \neq 0$, then one can assume $p_0 = 1$ without loss of generality. This will reduce the components of the adjoint vector $\mathbf{p}(t)$ and constants λ_i by a factor of p_0 and will not influence the values of control variables calculated by maximizing the Hamiltonian with respect to $\mathbf{u}$ according to Eq. (1.85), because the Hamiltonian (Eq. 1.83) depends on p_i and p_0 linearly.

To determine the control $\mathbf{u}_0(t)$ satisfying the maximum principle and to calculate the corresponding value of the optimization criterion of Eq. (1.81) it is necessary to consecutively execute the following steps:

Step 1. Using Eq. (1.85) for maximizing the Hamiltonian function with respect to the control vector components, represent the control vector $\mathbf{u}_0(t)$ as a function of variables $\mathbf{x}_{u_0}(t), \mathbf{p}^{u_0}(t)$, and t

$$\mathbf{u}_0(t)=\mathbf{u}_0(\mathbf{x}_{u_0}(t),\mathbf{p}^{u_0}(t),t). \tag{1.91}$$

Step 2. Substitute the expression of Eq. (1.91) for $\mathbf{u}_0(t)$ into Eqs. (1.72) and (1.84) as well as into the boundary condition of Eq. (1.88) (or (1.89)) and solve the resulting boundary-value problem for state variables $\mathbf{x}_{u_0}(t)$, adjoint variables $\mathbf{p}^{u_0}(t)$, and constants λ_i.

Step 3. Substitute the functions $\mathbf{x}_{u_0}(t)$ and $\mathbf{p}^{u_0}(t)$ into Eq. (1.91), thereby expressing the control $\mathbf{u}_0$ as an explicit function of time.

Step 4. Substitute the expressions for $\mathbf{x}_{u_0}(t)$ and $\mathbf{u}_0(t)$ into functions $\Phi_1(\mathbf{x},\mathbf{u},t)$ and $\Phi_2(\mathbf{x},t)$ and calculate the corresponding value of the optimization criterion according to Eq. (1.81).

Some final remarks on the maximum principle are appropriate. In general, the maximum principle is a necessary condition for optimality, not a sufficient condition. Therefore, the optimality of the

control determined on the basis of the maximum principle requires additional analysis, such as a comparison of the values of the optimization criterion related to different controls satisfying the maximum principle. However, if the optimal control exists and the maximum principle singles out only one function $\mathbf{u}_0(t)$, then this function is the desired optimal control.

To determine the optimal control from the maximum principle, in general, a non-linear boundary-value problem for a system of ordinary differential equations has to be solved. A closed-form solution of this problem is possible only in exceptional cases, and, in practice, it is necessary to resort to numerical methods, when calculating optimal controls. The maximum principle is the basis of several numerical methods for searching for the optimal control. However, the discussion of these methods is beyond the scope of this book. See Chernousko and Banichuk (1973), Moiseev (1975), and Fedorenko (1978) for further information.

1.2.2.7 Maximum-Type Functionals. Pontryagin's maximum principle outlined above is applicable directly to optimal control problems with the functionals of the form of Eq. (1.81) only. The particular case of the functional of Eq. (1.81) is the integral functional of Eq. (1.71). In practice, such functionals occur in the use of integral criteria, for example, in vibration isolation problems in which steady-state vibrational motion is optimized.

When transient motion is to be optimized, which is the case, in particular, for shock isolation, the functionals characterizing the peak response to the external disturbances are often considered to be more appropriate. These functionals have the form of Eq. (1.70). Some of the functionals of such a kind are given in Eqs. (1.42)– (1.44) and (1.46). Thus we arrive at the optimal control problems with *maximum-type functionals.* In particular, this is the case for Problem 1.5.

Optimal control problems with maximum-type functionals are more complicated mathematically than the problems with the integral-terminal functionals of Eq. (1.81). An analytical method, the efficiency of which could be compared with that of the maximum principle, is not available for the problems with maximum-type functionals. Therefore, it is reasonable to investigate the possibilities for reducing the optimal control problem with the functional of Eq. (1.70) to the problem with the functional of Eq. (1.81) or approximating the former problem by the latter one. One of these possibilities, based on the integral approximation of the maximum-type functionals, is outlined next.

1.2.2.8 Integral Approximation of Maximum-Type Functionals. The function $\Phi(\mathbf{x}, \mathbf{u}, t)$, for which the maximum as in Eq. (1.70) serves as a performance index of isolation, is usually nonnegative. Normally, this function represents the absolute value or square of some mechanical characteristic of the motion of the system (Section 1.1.3). If for any admissible control $\mathbf{u}(t)$, the function $\Phi(\mathbf{x}_u(t), \mathbf{u}(t), t)$ is nonnegative and continuous on the interval $[t_0, T]$, then the maximum-type functional of Eq. (1.70) and the integral functional

$$J_\mu(\mathbf{u}) = \int_{t_0}^{T} \Phi^\mu(\mathbf{x}_u(t), \mathbf{u}(t), t)dt \tag{1.92}$$

are related by

$$J(\mathbf{u}) = \max_{t\in[t_0,T]} \Phi(\mathbf{x}_u(t), \mathbf{u}(t), t) = \lim_{\mu\to\infty} [J_\mu(u)]^{1/\mu} =$$

$$\lim_{\mu\to\infty} \left[\int_{t_0}^{T} \Phi^\mu(\mathbf{x}_u(t), \mathbf{u}(t), t)dt\right]^{1/\mu}, \tag{1.93}$$

where μ is a positive numerical parameter.

Equation (1.93) can be used as the basis for solving optimal control problems with maximum-type functionals by using the maximum principle. The functional of Eq. (1.92) has the form of Eq. (1.81), where $\Phi_1 = \Phi^\mu$ and $\Phi_2 = 0$ and can be minimized with the use of the maximum principle. To each value of the parameter μ, there corresponds a piecewise continuous control $\mathbf{u}_0^\mu(t)$ that satisfies the constraint $\mathbf{u}_0^\mu(t) \in \mathbf{U}$ and minimizes the functional $J_\mu(\mathbf{u})$. Assume that for a particular metric introduced in the space of control functions, the sequence $\mathbf{u}_0^\mu$ converges to an admissible control, functionals $J_\mu(\mathbf{u})$ are continuous, and the convergence to the limit in Eq. (1.93) is uniform with respect to $\mathbf{u}$. Then the relationship

$$\lim_{\mu\to\infty} \mathbf{u}_0^\mu = \mathbf{u}_0 \tag{1.94}$$

is valid, where $\mathbf{u}_0$ is the optimal control minimizing the functional of Eq. (1.70). Thus, the optimal control in problems of minimizing the maximum of a function of state variables and time can be obtained as the limit of the sequence of admissible controls $\mathbf{u}_0^\mu$ minimizing integral functionals of the form of Eq. (1.92).

This method can be implemented on a computer by using available numerical methods for solving optimal control problems with integral functionals. See, for example, Chernousko and Banichuk (1973), Moiseev (1975), and Fedorenko (1978). In one application, this approach has been used by Johnson (1967) for solving the problem of minimizing the maximum square of the deviation of a particle from a specified position.

Other methods are also available for solving optimal control problems with maximum-type functionals that take into account specific features of such problems more thoroughly than the aforementioned approach does. However, the discussion of these methods is beyond the scope of this book. The interested reader can consult such literature as Vinogradova and Demyanov (1973, 1974), Silina (1976), and Timoshina and Shablinskaya (1980).

1.2.2.9 Graphical-Analytical Technique for Solving the Simplest Problem of Optimal Isolation. For some typical problems of optimal isolation, special methods of solution taking into account specific features of the problems are available. For example, for the simplest problem of optimal isolation (Problem 1.5) for a single-degree-of-freedom system governed by Eq. (1.74), a graphical-analytical procedure of calculating the optimal isolator characteristic as a function of time has been developed independently by Guretskii (1969) and Sevin and Pilkey (1971) for the case $T = \infty$. The shape of the optimal characteristic depends on the number of time intervals in which the absolute value of the external disturbance $v(t)$ exceeds the maximum admissible force

u_*, applied to the object being isolated by the isolator. To understand the fundamentals of the graphical-analytical approach, consider the simplest case where there is the only one interval (τ_1, τ_2) in which $| v(t) |> u_*$, i.e.

$$\begin{gathered} |v(t)| \leq u_* \quad \text{if} \quad 0 \leq t \leq \tau_1 \quad \text{or} \quad t \geq \tau_2, \\ |v(t)| > u_* \quad \text{if} \quad \tau_1 < t < \tau_2. \end{gathered} \tag{1.95}$$

In addition, assume that $v(t)$ is a continuous function and, without loss of generality, that $v(t) > u_*$ for $t \in (\tau_1, \tau_2)$. It is proved by Guretskii (1969) that if an optimal control in Problem 1.5 exists, then one can always choose the optimal control $u = u_0(t)$ so that $u_0 = u_*, u_0 = -u_*$ or $u_0 = v(t)$ in different time intervals.

If $\tau_1 = 0$, then the solution of the basic problem of optimal isolation is given by

$$\begin{gathered} u_0(t) = u_* \quad \text{if} \quad 0 \leq t \leq t_*, \qquad \dot{x}(t_*) = 0, \\ u_0(t) = v(t) \quad \text{if} \quad t \geq t_*. \end{gathered} \tag{1.96}$$

Here, t_* is the first non-zero time instant at which the velocity of the body being isolated relative to the base vanishes.

To prove this, represent the solution of the differential equation of Eq. (1.74), subject to the initial conditions of Eq. (1.75), in the form

$$x(t) = \int_0^t (t-\tau)[v(\tau) - u(\tau)]d\tau. \tag{1.97}$$

Since, according to the constraint, $u(t) \leq u_*$ for all t, the inequality

$$x(t) \geq \int_0^t (t-\tau)[v(\tau) - u_*)]d\tau \tag{1.98}$$

is valid. The assumption of $v(t) > u_*$ for $0 = \tau_1 < t < \tau_2$ implies that the right-hand side in Eq. (1.98) monotonically increases on the interval $0 \leq t < t_*$, where t_* is the point at which the derivative of the right-hand side of Eq. (1.98) vanishes, and is nonnegative on this interval. With allowance for this fact, the estimate

$$\max_t |x(t)| \geq \int_0^{t_*} (t_* - \tau)[v(\tau) - u_*)]d\tau \tag{1.99}$$

follows from Eq. (1.98). For the control of Eq. (1.96), the time history of x is

$$x(t) = \begin{cases} \int_0^t (t-\tau)[v(\tau) - u_*)]d\tau, & \text{if } 0 \leq t \leq t_* \\ \int_0^{t_*} (t_* - \tau)[v(\tau) - u_*)]d\tau = x(t_*), & \text{if } t > t_* \end{cases}. \tag{1.100}$$

It follows from Eqs. (1.99) and (1.100) that the lower bound in Eq. (1.99) is reached by the function $x(t)$ of Eq. (1.100). Hence, the control of Eq. (1.96) is optimal.

Consider now the case of $\tau_1 > 0$. Integrating Eq. (1.74) with the initial conditions of Eq. (1.75) yields

$$\dot{x}(t) = S(t) - U(t), \quad x(t) = \int_0^t S(\tau)d\tau - \int_0^t U(\tau)d\tau, \tag{1.101}$$

$$S(t) = \int_0^t v(\tau)d\tau, \qquad U(t) = \int_0^t u(\tau)d\tau.$$

Introduce the notation

$$A_i^U = \int_{t_{i-1}}^{t_i} S(\tau)d\tau - \int_{t_{i-1}}^{t_i} U(\tau)d\tau, \tag{1.102}$$

$$\dot{x}(t_i) = S(t_i) - U(t_i) = 0, \quad t_0 = 0, \quad i = 1, 2 \ldots,$$

where t_i are the times of intersection of plots of the functions $S(t)$ and $U(t)$ defined in Eq. (1.101). See Fig. 1.6. At instants t_i, the relative velocity $\dot{x}$ of the body being isolated vanishes. From the geometrical point of view, the value $\mid A_i^U \mid$ is equal to the area of the figure formed by the segments of the graphs of the functions $S(t)$ and $U(t)$ in the time interval $[t_{i-1}, t_i]$. If the difference $S(t) - U(t)$ changes its sign at a point $t = t_i$, then the function $x(t)$ has a local extremum at the point t_i. It follows from Eqs. (1.101) and (1.102) that

$$x_u(t_i) = \sum_{k=1}^{i} A_k^U. \tag{1.103}$$

Equations (1.101)–(1.103) imply the following single-valued correspondence between the quantities $S(t), U(t)$, and A_i^U, on the one hand, and $v(t), u(t)$, and $\max_t |x_u(t)|$, on the other hand,

$$v(t) = \frac{dS(t)}{dt}, \quad u(t) = \frac{dU(t)}{dt}, \quad \max_{t \in [0,\infty)} |x_u(t)| = \max_{i \geq 1} \left| \sum_{k=1}^{i} A_k^U \right|. \tag{1.104}$$

Taking into account Eq. (1.104), one can reformulate Problem 1.5 in terms of the quantities $S(t), U(t)$, and A_i^U as follows.

1.2.2.10 <u>Problem 1.6</u>. Given the function $S(t)$, find a piecewise differentiable function $U_0(t)$, differentiable on the right at each point of its domain, such that

$$\max_{i \geq 1} \left| \sum_{k=1}^{i} A_i^{U_0} \right| = \min_U \max_{i \geq 1} \left| \sum_{k=1}^{i} A_i^U \right|, \quad \left| \dot{U}(t) \right| \leq u_*. \tag{1.105}$$

This formulation of the simplest problem of optimal isolation is convenient for a graphical solution.

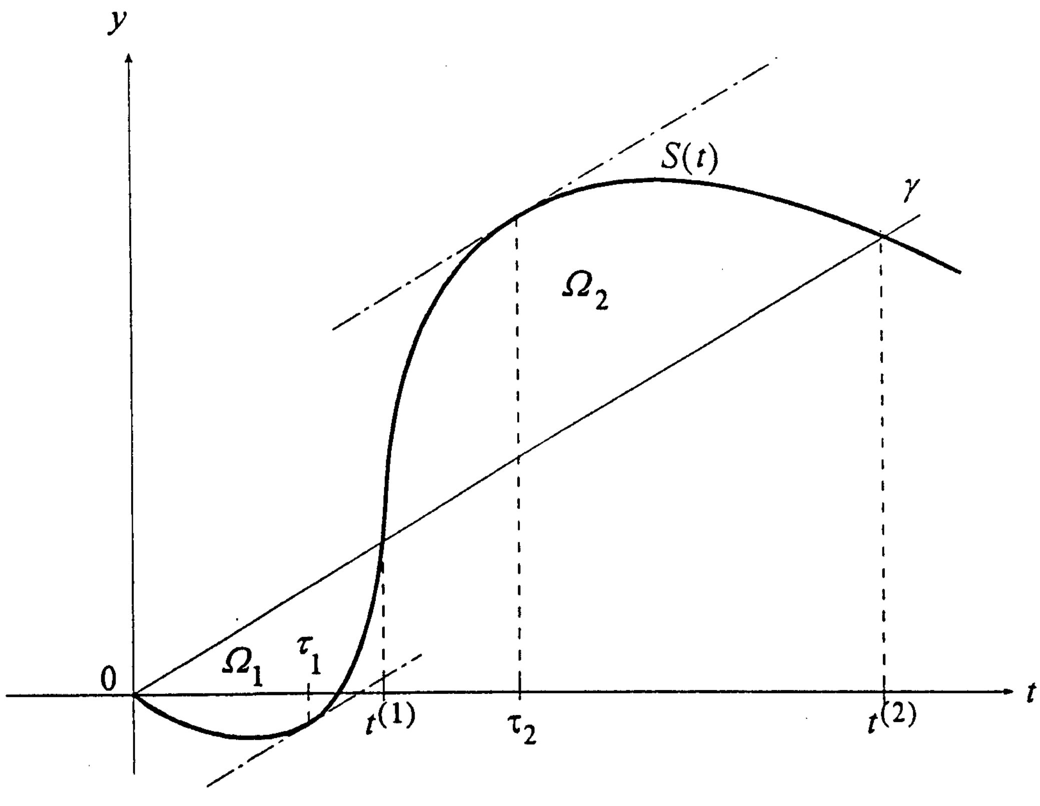


Figure 1-6. The relative position of the curves $S(t)$ and γ.

The shape of the function $U_0(t)$ depends on the relative arrangement of the graph of the function $y = S(t)$ and the straight line γ defined by the equation $y = u_* t$. It is evident from the inequality in Eq. (1.105) that the tangent of the inclination angle of this line to the time axis is equal to the maximum admissible value u_* of the tangent of the inclination angle of the line tangential to the graph of the function $y = U(t)$. The cases considered below give complete solutions to the simplest problem of optimal isolation.

Case 1. The line γ intersects the plot of the function $y = S(t)$ as shown in Fig. 1.6. The properties of Eq. (1.95) of the function $v(t)$ and the definition of the function $S(t)$ imply that the line γ can intersect the graph of the function $y = S(t)$ only at two points, except the origin. Let us denote the abscissae of these points by $t^{(1)}$ and $t^{(2)}$ $(t^{(1)} < t^{(2)})$, and the areas of the figures formed by the segments of the line γ and the plot of the function $y = S(t)$ in time intervals $[0, t^{(1)}]$ and $[t^{(1)}, t^{(2)}]$ by Ω_1 and Ω_2, respectively.

Case 1.1. $\Omega_2 > 2\Omega_1$. If this inequality holds, then the function $U_0(t)$ of Problem 1.6 is given by

$$U_0(t) = u_* t \quad \text{if} \quad 0 \le t < t^{(2)}; \qquad U_0(t) = S(t) \quad \text{if} \quad t \ge t^{(2)}. \tag{1.106}$$

In this case

$$t_1 = t^{(1)}, \quad t_2 = t^{(2)}, \quad A_1^{U_0} = -\Omega_1, \quad A_2^{U_0} = \Omega_2,$$

$$\max_{i \ge 1} \left| \sum_{k=1}^{i} A_i^{U_0} \right| = \Omega_2 - \Omega_1 > \Omega_1. \tag{1.107}$$

According to Eqs. (1.104) and (1.106), in the case of $\Omega_2 > 2\Omega_1$, the optimal isolator characteristic $u_0(t)$ of Problem 1.5 and the corresponding maximum absolute value of the displacement of the body being isolated are given by

$$u_0(t) = u_* \quad \text{if} \quad 0 \le t < t^{(2)}; \qquad u_0(t) = v(t) \quad \text{if} \quad t \ge t^{(2)}; \tag{1.108}$$

$$\max_{t \in [0,\infty)} |x_{u_0}(t)| = x_{u_0}(t^{(2)}) = \Omega_2 - \Omega_1.$$

Let us show that the function $U_0(t)$ in Eq. (1.106) indeed gives a solution to Problem 1.6. Consider an arbitrary function $U'(t) \not\equiv U_0(t)$ satisfying the constraint of Eq. (1.105). By virtue of this constraint, the plot of this function does not lie above the line γ. The properties of the function $v(t)$ given in Eq. (1.95) imply that the plots of the functions $U'(t)$ and $S(t)$ can intersect only in the intervals $[0, t^{(1)}]$ and $[t^{(2)}, \infty)$.

Denote the abscissa of the first intersection point of the plots of the functions $S(t)$ and $U'(t)$ in the interval $[t^{(2)}, \infty)$ by $\tilde{t}$. It may turn out that the plots of the functions $S(t)$ and $U'(t)$ do not

intersect at $t \geq t^{(2)}$. In this case, formally set $\widetilde{t} = \infty$. If the plot of the function $U'(t)$ in the interval $(0, t^{(1)}]$ lies below the plot of the function $S(t)$, as shown in Fig. 1.7, then

$$\widetilde{t} = t_1, \quad x(\widetilde{t}) = x(t_1) = A_1^{U'} > \Omega_2 > \Omega_2 - \Omega_1. \tag{1.109}$$

Let the plots of the functions $U'(t)$ and $S(t)$ in the interval $(0, t^{(1)}]$ intersect at the points with abscissae $t = t_i, \quad i = 1, .., n$, where n is a positive integer. In this case (Fig. 1.8), $\widetilde{t} = t_{n+1}$ and

$$x(\widetilde{t}) = \sum_{i=1}^{n+1} A_i^{U'} = \sum_{k \in I_+} A_k^{U'} + \sum_{s \in I_-} A_s^{U'} + A_{n+1}^{U'}, \tag{1.110}$$

$$\sum_{k \in I_+} A_k^{U'} > 0, \qquad 0 < -\sum_{s \in I_-} A_s^{U'} < \Omega_1, \qquad A_{n+1}^{U'} > \Omega_2$$

By I_+ and I_- in Eq. (1.110) we denote the sets of indices $i = 1, \ldots, n$ such that $A_i^{U'} > 0$ for $i \in I_+$ and $A_i^{U'} < 0$ for $i \in I_-$. It follows directly from Eq. (1.110) that

$$x(\widetilde{t}) = x(t_{n+1}) > \Omega_2 - \Omega_1. \tag{1.111}$$

Equations (1.108), (1.109), and (1.111) imply that any function $U'(t) \not\equiv U_0(t)$ satisfying the constraint of Eq. (1.105) corresponds to a larger value of the maximum modulus of the displacement of the body being isolated, compared with that for the function $U_0(t)$. This proves the optimality of the function $U_0(t)$.

Case 1.2. $\Omega_1 < \Omega_2 \leq 2\Omega_1$. If these inequalities hold, then the function $U_0(t)$ of Problem 1.6 is given by

$$\begin{aligned} U_0(t) &= -u_* t \quad \text{if} \quad 0 \leq t < \tfrac{\delta}{2u_*}; \\ U_0(t) &= u_* t - \delta \quad \text{if} \quad \tfrac{\delta}{2u_*} \leq t < t_3; \\ U_0(t) &= S(t) \quad \text{if} \quad t \geq t_3, \end{aligned} \tag{1.112}$$

where t_3 denotes the abscissa of the first intersection point of the straight line $y = u_* t - \delta$ ($\delta > 0$) and the plot of function $S(t)$ to the right of $t^{(2)}$. The shape of the function $U_0(t)$ is shown in Fig. 1.9. The parameter $\delta > 0$ in Eq. (1.112) is chosen in such a way that the plots of the functions $S(t)$ and $U_0(t)$ intersect at three points, apart from the origin, with the abscissae $t = t_1$, $t = t_2$, and $t = t_3$. Moreover, the parameter δ is chosen so that

$$A_3^{U_0} = -2(A_1^{U_0} + A_2^{U_0}). \tag{1.113}$$

Such a value of the parameter δ exists because the quantities $A_i^{U_0}$, $i = 1, 2, 3$, depend on δ continuously, the limiting relation $A_3^{U_0} + 2(A_1^{U_0} + A_2^{U_0}) \to \Omega_2 - 2\Omega_1 \leq 0$ is valid as $\delta \to 0$, and there exists $\delta > 0$ (for example, corresponding to tangency of the plot of the function $S(t)$ by the line $y = u_* t - \delta$ at a point with the abscissa $t \in (0, t^{(1)})$) such that $A_3^{U_0} + 2(A_1^{U_0} + A_2^{U_0}) > 0$.

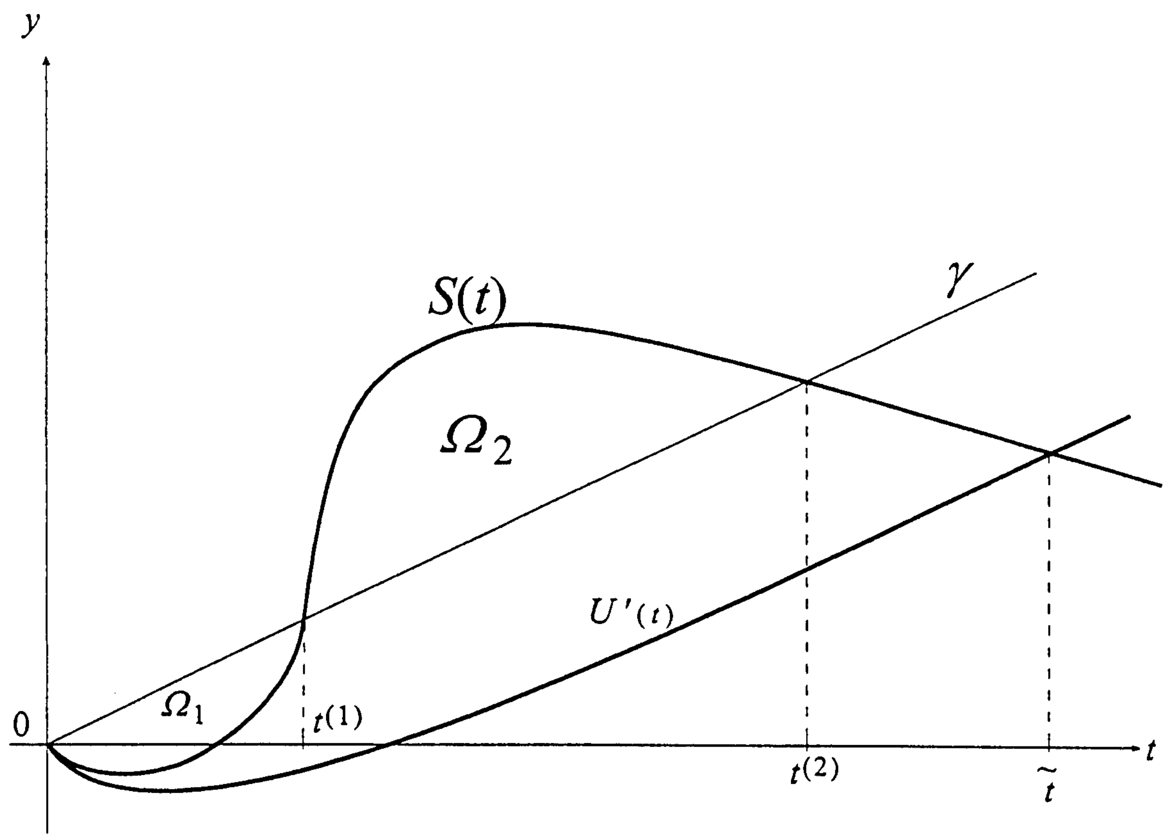


Figure 1-7. The relative position of the curves $S(t)$ and $U'(t)$ in the case where these curves do not intersect in the interval $[0, t^{(1)}]$.

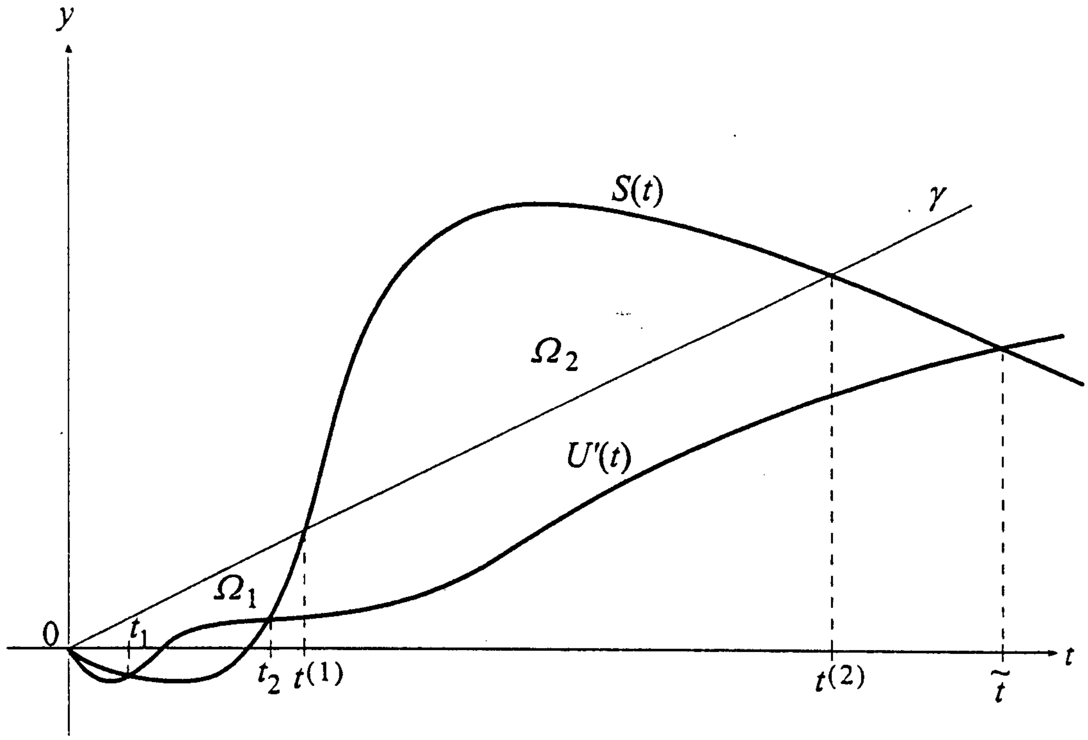


Figure 1-8. The relative position of the curves $S(t)$ and $U'(t)$ in the case where these curves have intersection points in the interval $[0, t^{(1)}]$.

We choose to omit the proof of the optimality of the function $U_0(t)$ given by Eq. (1.112). In many aspects, it is analogous to the proof of the optimality for the case of $\Omega_2 > 2\Omega_1$. Details of the proof are given in such publications as Guretskii (1969a) and Kolovskii (1976).

The optimal isolator characteristic $u_0(t)$ corresponding to Eq. (1.112) is expressed as

$$\begin{array}{rl} u_0(t) = -u_* & \text{if} \quad 0 \le t < \frac{\delta}{2u_*}; \\ u_0(t) = u_* & \text{if} \quad \frac{\delta}{2u_*} \le t < t_3; \\ u_0(t) = v(t) & \text{if} \quad t \ge t_3. \end{array} \tag{1.114}$$

It is seen in Fig. 1.9 that

$$A_1^{U_0} > 0, \quad \left|A_2^{U_0}\right| = -A_2^{U_0} < \Omega_1, \quad \left|A_3^{U_0}\right| = A_3^{U_0} > \Omega_2. \tag{1.115}$$

Provided $\Omega_1 < \Omega_2$, Eq. (1.115) implies that $-A_3^{U_0} < A_2^{U_0}$. Therefore, if the parameter δ in Eq. (1.112) is chosen so that Eq. (1.113) holds, then the relationship

$$A_1^{U_0} = -A_3^{U_0}/2 - A_2^{U_0} < -A_2^{U_0}/2 \tag{1.116}$$

is valid. It follows from Eqs. (1.113) and (1.116) that the maximum absolute value of the displacement of the body being isolated corresponding to the optimal isolator characteristic $u_0(t)$ of Eq. (1.114) is given by

$$\begin{array}{c} \max_{t\in[0,\infty)} |x_{u_0}(t)| = -x_{u_0}(t_2) = -A_1^{U_0} - A_2^{U_0} = x_{u_0}(t_3) = \\ A_1^{U_0} + A_2^{U_0} + A_3^{U_0}. \end{array} \tag{1.117}$$

Case 1.3. $\Omega_2 \le \Omega_1$. If this inequality holds, then the desired optimal function $U_0(t)$ is determined as follows. Construct the function $\widetilde{U}(t)$ which has the form analogous to Eq. (1.112)

$$\begin{array}{rl} \widetilde{U}(t) = -u_* t & \text{if} \quad 0 \le t < \frac{\beta}{2u_*}; \\ \widetilde{U}(t) = u_* t - \beta & \text{if} \quad \frac{\beta}{2u_*} \le t < t_3; \\ \widetilde{U}(t) = S(t) & \text{if} \quad t \ge t_3. \end{array} \tag{1.118}$$

The parameter β in Eq. (1.118) is chosen so that the graphs of the functions $S(t)$ and $\widetilde{U}(t)$ intersect at three points, except the origin, with the abscissae t_1, t_2, and t_3 $(t_1 < t_2 < t_3)$ and the quantities $A_2^{\widetilde{U}}$ and $A_3^{\widetilde{U}}$ are related by $\mid A_2^{\widetilde{U}} \mid = -A_2^{\widetilde{U}} = A_3^{\widetilde{U}}$. Such a value of the parameter β exists if $\Omega_2 \le \Omega_1$.

If $0 < A_1^{\widetilde{U}} < -A_2^{\widetilde{U}}/2$, then the optimal function $U_0(t)$ is specified by Eqs. (1.112) and (1.113), while the corresponding optimal isolator characteristic u_0 and the minimum of the maximum absolute value of the displacement of the body being isolated are determined by Eqs. (1.114) and (1.117), respectively.

If $A_1^{\widetilde{U}} \ge -A_2^{\widetilde{U}}/2$, then the optimal function $U_0(t)$ is specified as follows:

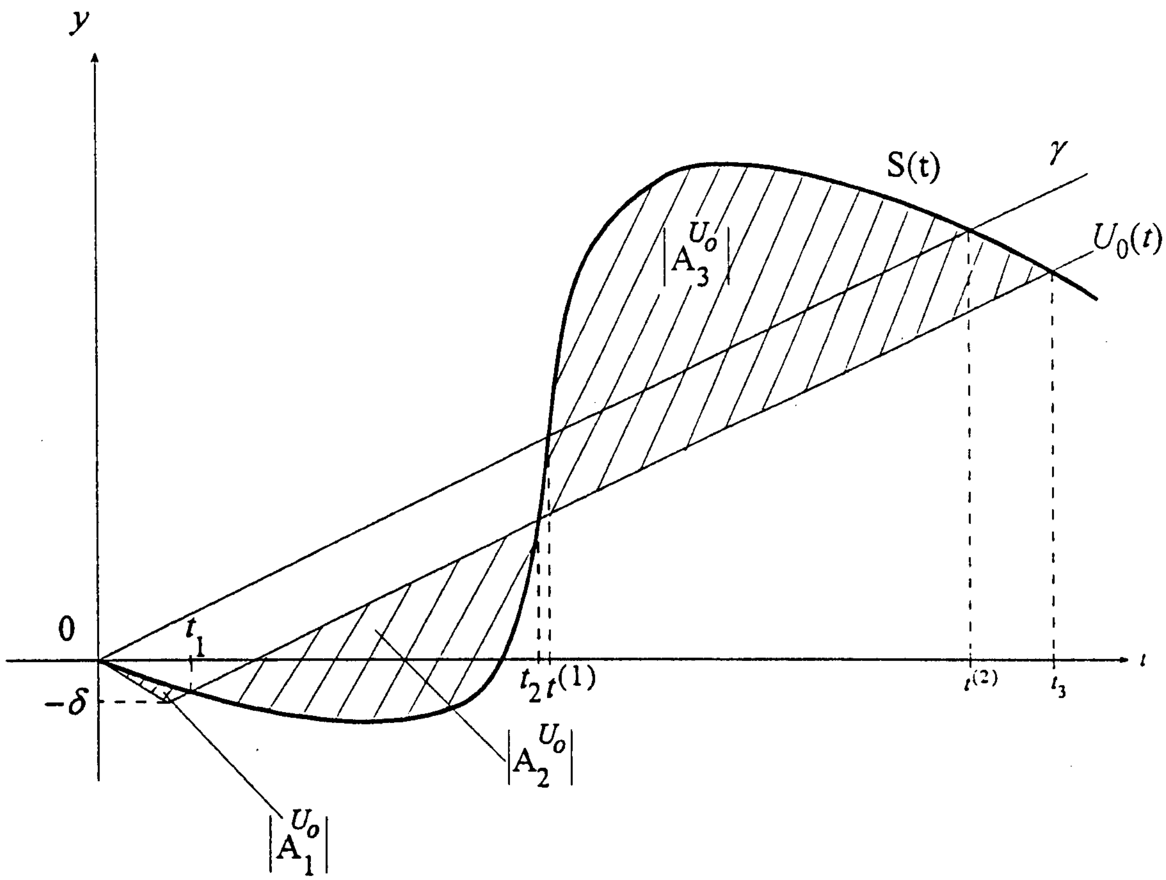


Figure 1-9. The function $U_0(t)$.

$$U_0(t) = \xi(t) \quad \text{if} \quad 0 \le t < \frac{\beta}{2u_*}; \qquad U_0(t) = \widetilde{U}(t) \quad \text{if} \quad t \ge \frac{\beta}{2u_*}. \tag{1.119}$$

Here, $\xi(t)$ is an arbitrary piecewise differentiable function defined over the interval $0 \le t \le \beta/(2u_*)$, differentiable on the right at any point of the interval $0 < t < \beta/(2u_*)$ and providing the relationships $A_1^{U_0} = -A_2^{U_0}/2 = -A_2^{\widetilde{U}}/2$. Such a function exists, because inside the figure bounded by the graphs of the functions $\widetilde{U}(t)$ and $S(t)$ in the interval $[0, t_1]$ and having the area equal to $A_1^{\widetilde{U}} > \left|A_2^{\widetilde{U}}\right|/2$, one can always draw a curve $y = \xi(t)$ passing through the origin and the point with the abscissa $t = t_1$ so that the area of the figure bounded by the graphs of the functions $\xi(t)$ and $S(t)$ in the interval $[0, t_1]$ is equal to any prescribed positive number $A \le A_1^{\widetilde{U}}$. In particular, the function $\xi(t)$ can be chosen to be piecewise linear, with the derivative being equal to $-u_*$ or u_*.

We omit the proof of the optimality of the function of Eq. (1.119), which can be found in detail in Guretskii (1969a).

The function $U_0(t)$ in Eq. (1.119) corresponds to the optimal isolator characteristic $u_0(t)$ given by

$$\begin{aligned} u_0(t) &= \dot{\xi}(t) \quad \text{if} \quad 0 \le t < t_1; \\ u_0(t) &= u_* \quad \text{if} \quad t_1 \le t < t_3; \\ u_0(t) &= v(t) \quad \text{if} \quad t \ge t_3. \end{aligned} \tag{1.120}$$

Note that generally the function $\xi(t)$ is piecewise differentiable. At the points of nondifferentiability, $\dot{\xi}(t)$ should be understood to be the corresponding derivative on the right. If the function $\xi(t)$ is chosen to be piecewise linear, with the derivative being equal to $-u_*$ or u_*, then the optimal isolator characteristic of Eq. (1.120) takes on the values $-u_*$or u_* in the interval $0 \le t < t_3$ and is equal to $v(t)$ for $t \ge t_3$. The maximum absolute value of the displacement of the body being isolated corresponding to the optimal isolator characteristic of Eq. (1.120) is given by

$$\max_{t \in [0,\infty)} |x_{u_0}(t)| = x_{u_0}(t_1) = A_1^{U_0} = \left|A_2^{U_0}\right|/2 = -x_{u_0}(t_2) = x_{u_0}(t_3). \tag{1.121}$$

Case 2. The line γ intersects the plot of the function $y = S(t)$ at no points, except the origin $(t = 0, y = 0)$.

Is this case the optimal function $U_0(t)$ of Problem 1.6 is constructed quite analogously to case 1.3 and is defined by Eq. (1.112) if $A_1^{\widetilde{U}} < -A_2^{\widetilde{U}}/2$, and by Eq. (1.119) if $A_1^{\widetilde{U}} \ge -A_2^{\widetilde{U}}/2$.

The article by Guretskii (1969a) devoted to the graphical-analytical approach to the solution of Problem 1.5 presents the algorithm for constructing the optimal isolator characteristic in the case of an arbitrary number of time intervals in which the absolute value of the disturbance exceeds the maximum admissible force applied to the body being isolated.

The major advantage of the graphical-analytical approach is that in some cases it allows estimating the structure of the optimal isolator characteristic (in particular, the number of the

switch points) without cumbersome calculations. Note, however, that the graphical-analytical procedure is applicable only to single-degree-of-freedom systems governed by the second-order scalar differential equation of Eq. (1.74).

1.2.2.11 Solution of the Problem of Optimal Isolation for a Half-Sine Disturbance. Solve Problem 1.5 using the graphical-analytical technique for an external disturbance in the form of a half-sine pulse, that is, the excitation is given by

$$v(t) = \begin{cases} a \sin \frac{\pi t}{T_*}, & \text{for } 0 \le t \le T_* \\ 0, & \text{for } t > T_* \end{cases}, \tag{1.122}$$

where a is the amplitude of the pulse and T_* is its duration. According to Eqs. (1.74) and (1.75), the motion of the body to be isolated is governed by the differential equation with zero initial conditions

$$\ddot{x} + u = v(t), \qquad x(0) = 0, \quad \dot{x}(0) = 0. \tag{1.123}$$

We consider the problem for an infinite time interval and set $T = \infty$ in Eqs. (1.76) to (1.79). Accordingly, the problem is reduced to the minimization of the functional

$$J_1(u) = \max_{t \in [0,\infty)} |x_u(t)|, \tag{1.124}$$

where $x_u(t)$ is the solution of the initial value problem of Eq. (1.123), subject to the constraint

$$|u(t)| \le u_*. \tag{1.125}$$

1.2.2.12 Dimensionless Variables. The problem to be solved contains three independent parameters, namely, a, T_*, and u_*. Introduce the dimensionless (primed) variables

$$x' = \frac{\pi^2}{u_* T_*^2} x, \quad t' = \frac{\pi t}{T_*}, \quad a' = \frac{a}{u_*}, \quad u' = \frac{u}{u_*}, \quad v' = \frac{v}{u_*}, \quad J_1' = \frac{\pi^2}{u_* T_*^2} J_1. \tag{1.126}$$

into Eqs. (1.122) to (1.125). The disturbance of Eq. (1.122) becomes

$$v(t) = \begin{cases} a \sin t, & \text{for } 0 \le t \le \pi \\ 0, & \text{for } t > \pi \end{cases} \tag{1.127}$$

and the constraint of Eq. (1.125) acquires the form

$$|u(t)| \le 1. \tag{1.128}$$

The primes indicating the dimensionless variables are omitted for convenience. The form of Eqs. (1.123) and (1.124) remains unchanged under the change of variables of Eq. (1.126).

Note that the equations expressed with dimensionless variables have only one parameter a, instead of three parameters. This simplifies the solution and analysis of the problem. If necessary, one may return to the original dimensional variables using the formulas of Eq. (1.126).

1.2.2.13 Construction of the Optimal Control. Construct the optimal control for $a > 1$. For $a \leq 1$, the solution to the problem is trivial. By setting $u(t) \equiv v(t)$ we provide $x_u(t) \equiv 0$ and, hence, $J_1(u) = 0$. It is seen from Eq. (1.127) that $|v(t)| \leq 1$ if $a \leq 1$ and, hence, the constraint of Eq. (1.125) is satisfied.

For $a > 1$, we have

$$\begin{aligned} v(t) &\geq 0, && \text{for all} \quad t; \\ v(t) &< 1, && \text{for} \quad 0 \leq t < \tau_1 \quad \text{and} \quad t > \tau_2; \\ v(t) &> 1, && \text{for} \quad \tau_1 < t < \tau_2, \end{aligned} \tag{1.129}$$

where τ_1 and τ_2 are the roots of the equation

$$a \sin t = 1 \tag{1.130}$$

on the interval $0 \leq t \leq \pi$. Solve Eq. (1.130) for τ_1 and τ_2 to find

$$\tau_1 = \sin^{-1}(\frac{1}{a}), \quad \tau_2 = \pi - \sin^{-1}(\frac{1}{a}). \tag{1.131}$$

Hence, the disturbance $v(t)$ of Eq. (1.127) belongs to the class of disturbances of Eq. (1.95), and the graphical-analytical technique described previously can be applied.

According to this technique, the function $S(t)$ of Eq. (1.101) and the line $\gamma = u_* t$ should be plotted on a single coordinate plane. In our case, we have

$$S(t) = \int_0^t v(\tau) d\tau = \begin{cases} a(1 - \cos t), & \text{for } 0 \leq t \leq \pi \\ 2a, & \text{for } t > \pi \end{cases}, \tag{1.132}$$

$$\gamma = t. \tag{1.133}$$

A study of the function

$$F(t) = S(t) - t \tag{1.134}$$

shows that $F(0) = 0$ for any a and, hence, the plots of $S(t)$ and γ always intersect at the coordinate origin. If $a < a_* \approx 1.380$, then this is the only intersection point (Fig. 1.10). If $a = a_*$, then the line γ intersects the plot of $S(t)$ at two points, one of which is the coordinate origin, and the other is the tangent point at $t = \tau_2 = \pi - \sin^{-1}(1/a_*) \approx 2.331$ (Fig. 1.11). If $a > a_*$, then the

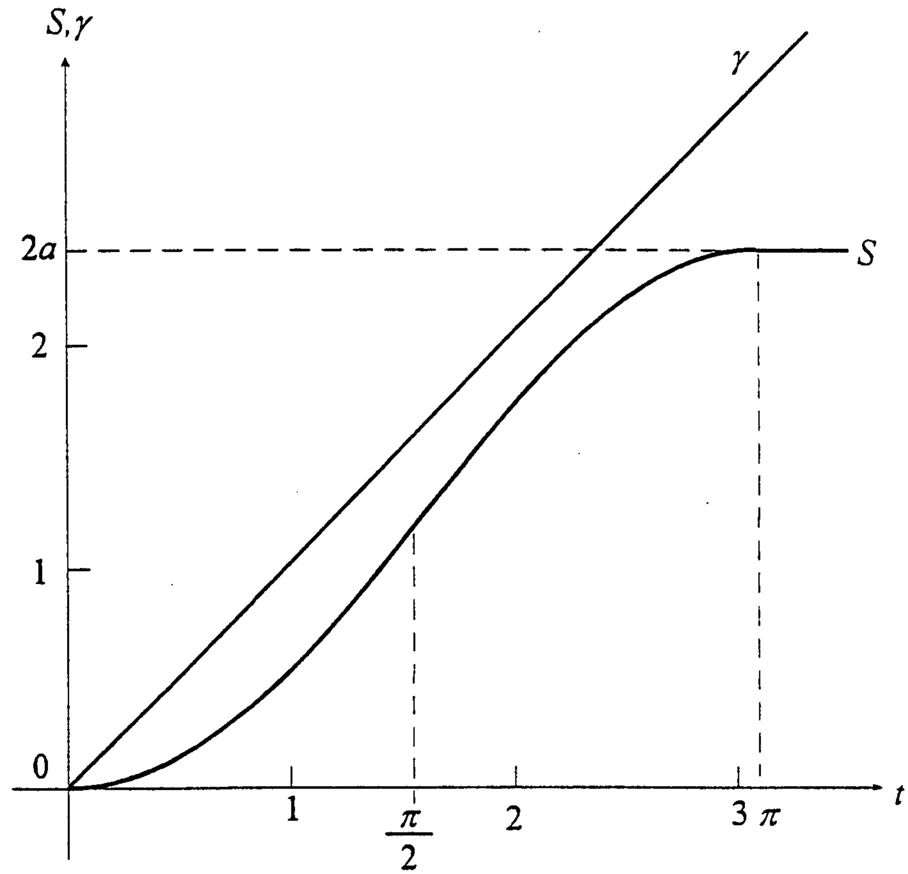


Figure 1-10. The relative position of the curve S and the line γ for $a<a_*$.

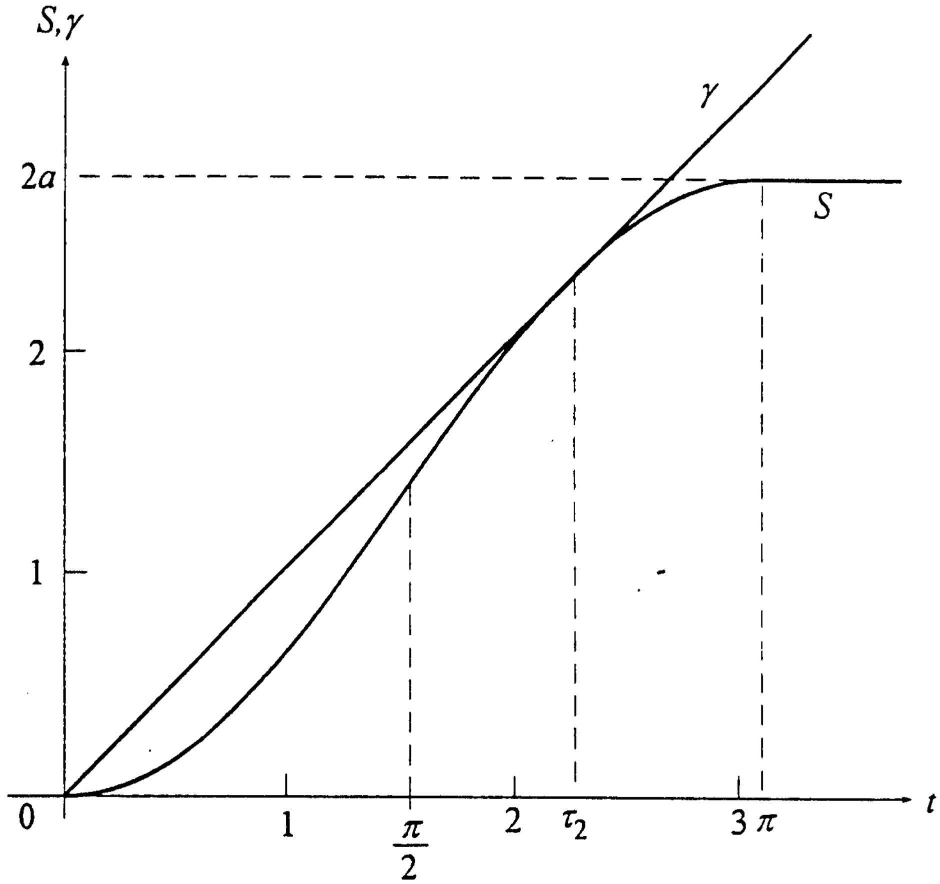


Figure 1-11. The relative position of the curve S and the line γ for $a = a_*$.

line γ intersects the plot of $S(t)$ at two points $t^{(1)}$ and $t^{(2)}$, $t^{(2)} > t^{(1)}$, other than the origin (Fig. 1.12). The quantity a_* is the root of the equation

$$a + \sqrt{a^2 - 1} - \pi + \sin^{-1}(1/a) = 0, \tag{1.135}$$

which follows from the tangency condition for the functions $S(t)$ and $\gamma = t$.

Case $a > a_*$. The points $t^{(1)}$ and $t^{(2)}$ of intersection of the curves $S(t)$ and γ for $a > a_*$ can be found as the roots of the function $F(t)$ of Eq. (1.134). The analysis of this function shows that if $a < \pi/2$, then both $t^{(1)}$ and $t^{(2)}$ lie in the interval $(0, \pi)$. In this case both of the roots can be found by numerical solution of the transcendental equation

$$a(1 - \sin t) - t = 0. \tag{1.136}$$

If $a \geq \pi/2$, then the second root $t^{(2)}$ corresponds to the intersection of the line γ with the rectilinear part of the plot of $S(t)$ for $t > \pi$. In this case, it follows that $t^{(2)} = 2a$.

Calculate now the areas Ω_1 and Ω_2 of the domains confined between the curves $S(t)$ and γ on the intervals $0 \leq t \leq t^{(1)}$ and $t^{(1)} \leq t \leq t^{(2)}$, respectively. We obtain

$$\Omega_1 = -\int_0^{t^{(1)}} F(\tau)d\tau = \frac{\left[t^{(1)}\right]^2}{2} - a[t^{(1)} - \sin t^{(1)}], \tag{1.137}$$

$$\Omega_2 = \int_{t^{(1)}}^{t^{(2)}} F(\tau)d\tau = \tag{1.138}$$

$$\begin{cases} a[t^{(2)} - \sin t^{(2)}] - a[t^{(1)} - \sin t^{(1)}] + \left[t^{(1)}\right]^2/2 - \left[t^{(2)}\right]^2/2, & \text{for } a < \pi/2 \\ -a[t^{(1)} - \sin t^{(1)}] + \left[t^{(1)}\right]^2/2 + a(2a - \pi), & \text{for } a \geq \pi/2 \end{cases}.$$

It can be shown that the difference

$$\Delta\Omega = \Omega_2 - \Omega_1 = \begin{cases} a\left[t^{(2)} - \sin t^{(2)}\right] - \left[t^{(2)}\right]^2/2, & \text{for } a < \pi/2 \\ a(2a - \pi), & \text{for } a \geq \pi/2 \end{cases} \tag{1.139}$$

increases monotonically as a increases. In particular,

$$\begin{aligned} \Delta\Omega &< 0, \quad \text{for} \quad a < \pi/2; \\ \Delta\Omega &= 0, \quad \text{for} \quad a = \pi/2; \\ \Delta\Omega &= a(2a - \pi) > 0, \quad \text{for} \quad a > \pi/2. \end{aligned} \tag{1.140}$$

Consider first the case where $\Omega_2 \geq 2\Omega_1$. According to Eq. (1.140), this inequality holds only if $a > \pi/2$, and, hence, $t^{(2)} = 2a > \pi$. In this case, the solution of the problem is given by Eq. (1.108), and, accordingly, the optimal control has the form

$$u_0 = \begin{cases} 1, & \text{for } 0 \leq t < 2a \\ 0, & \text{for } t \geq 2a \end{cases}. \tag{1.141}$$

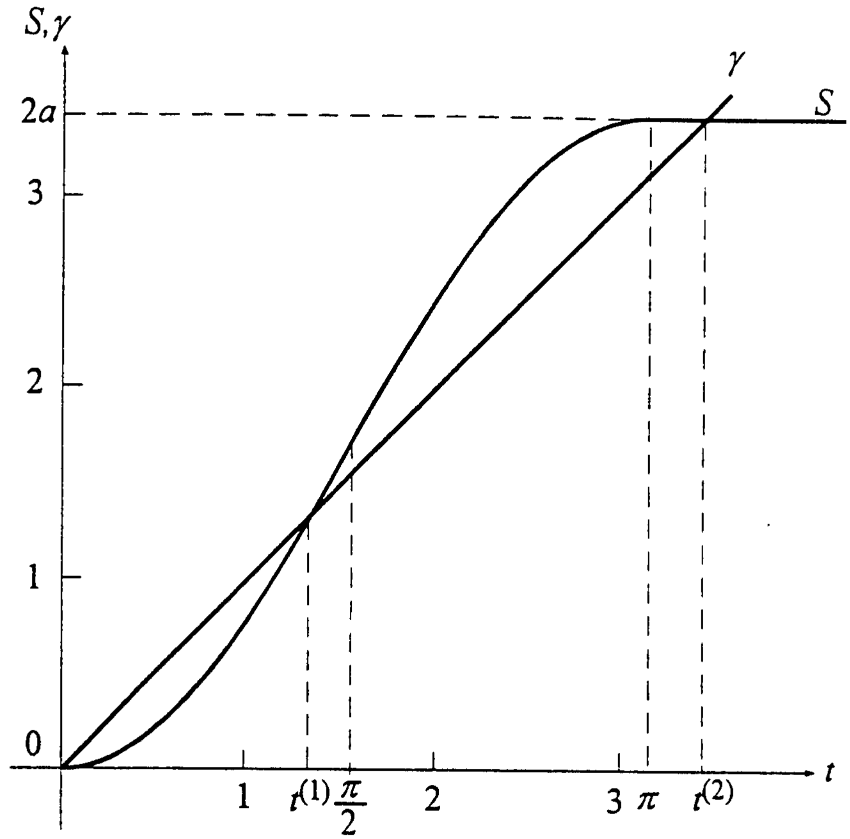


Figure 1-12. The relative position of the curve S and the line γ for $a > a_*$.

The corresponding minimum of the peak displacement is

$$J_1(u_0) = \Delta\Omega = a(2a - \pi). \tag{1.142}$$

Consider now the case where $\Omega_2 < 2\Omega_1$. Then the optimal control is given by the formulas of Eqs. (1.114) or (1.120), depending on the relationship between the quantities $A_1^{\bar{U}}, A_2^{\bar{U}}$, and $A_3^{\bar{U}}$, calculated according to Eqs. (1.102) and (1.118). In our case, these quantities are expressed as

$$A_1^{\bar{U}} = a(t_1 - \sin t_1) + \frac{\beta^2}{4} - \frac{1}{2}(t_1 - \beta)^2, \tag{1.143}$$

$$A_2^{\bar{U}} = -a(t_1 - \sin t_1) + a(t_2 - \sin t_2) - \frac{1}{2}(t_2 - \beta)^2 + \frac{1}{2}(t_1 - \beta)^2, \tag{1.144}$$

$$A_3^{\bar{U}} = a(t_3 - \sin t_3) - a(t_2 - \sin t_2) - \frac{1}{2}(t_3 - \beta)^2 + \frac{1}{2}(t_2 - \beta)^2. \tag{1.145}$$

Here, t_1, t_2, and t_3 are the coordinates of the intersection points of the plot of $S(t)$ and the line $\gamma_1 = t - \beta$, that is, three consecutive roots of the equation

$$S(t) - t + \beta = 0, \tag{1.146}$$

where $S(t)$ is defined in Eq.(1.132). It is evident from Eq. (1.146) that the roots t_1, t_2,and t_3 depend on the parameter β. To indicate this dependence we will write $t_1(\beta), t_2(\beta)$, and $t_3(\beta)$. Accordingly, the quantities $A_1^{\bar{U}}, A_2^{\bar{U}}$,and $A_3^{\bar{U}}$ of Eqs. (1.143) to (1.145) also depend on β, and we will write $A_1^{\bar{U}}(\beta), A_2^{\bar{U}}(\beta)$,and $A_3^{\bar{U}}(\beta)$. For $a > 1$, there exists the range of β such that Eq. (1.146) has three roots. This follows from the geometric properties of the plot of $S(t)$ and the line $\gamma_1 = t - \beta$. In practice, the transcendental equation of Eq. (1.146) has to be solved numerically.

To construct the optimal control for the case of $\Omega_1 \leq \Omega_2 \leq 2\Omega_1$ solve the equation

$$2\left[A_1^{\bar{U}}(\delta) + A_2^{\bar{U}}(\delta)\right] + A_3^{\bar{U}}(\delta) = 0 \tag{1.147}$$

for δ. Then the optimal control is given by Eq. (1.114) in which one should set $u_* = 1$. Accordingly, we have

$$u_0(t) = \begin{cases} -1, & \text{for } 0 \leq t < \delta/2 \\ 1, & \text{for } \delta/2 \leq t < t_3(\delta) \\ v(t), & \text{for } t \geq t_3(\delta) \end{cases}. \tag{1.148}$$

The minimum peak displacement corresponding to the optimal control of Eq. (1.148) is calculated according to Eq. (1.117). Then

$$J_1(u_0) = A_1^{U_0} + A_2^{U_0} + A_3^{U_0} = A_1^{\bar{U}}(\delta) + A_2^{\bar{U}}(\delta) + A_3^{\bar{U}}(\delta). \tag{1.149}$$

To construct the optimal control for the case of $\Omega_2 < \Omega_1$, one should execute the following operations.

1. Find the parameter β such that

$$A_2^{\tilde{U}}(\beta) + A_3^{\tilde{U}}(\beta) = 0. \tag{1.150}$$

2. If $A_1^{\tilde{U}}(\beta) + A_2^{\tilde{U}}(\beta)/2 \leq 0$, then solve the equation

$$2\left[A_1^{\tilde{U}}(\delta) + A_2^{\tilde{U}}(\delta)\right] + A_3^{\tilde{U}}(\delta) = 0 \tag{1.151}$$

for δ. The optimal control in this case is given by Eq. (1.114) in which one should set $u_* = 1$. Accordingly, we have

$$u_0(t) = \begin{cases} -1, & \text{for } 0 \leq t < \delta/2 \\ 1, & \text{for } \delta/2 \leq t < t_3(\delta) \\ v(t), & \text{for } t \geq t_3(\delta) \end{cases}. \tag{1.152}$$

The minimum peak displacement corresponding to the optimal control of Eq. (1.152) is calculated according to Eq. (1.117). Then

$$J_1(u_0) = A_1^{U_0} + A_2^{U_0} + A_3^{U_0} = A_1^{\tilde{U}}(\delta) + A_2^{\tilde{U}}(\delta) + A_3^{\tilde{U}}(\delta). \tag{1.153}$$

Note that the function U_0 of Eq. (1.112) coincides with $\tilde{U}$ of Eq. (1.118) for $\beta = \delta$.

3. If $A_1^{\tilde{U}}(\beta) + A_2^{\tilde{U}}(\beta)/2 > 0$, then construct the function $\xi(t)$ indicated in Eq. (1.119). The function $\xi(t)$ must be constructed so that

$$\int_0^{t_1(\beta)} [S(t) - \xi(t)]dt = A_2^{\tilde{U}}(\beta)/2. \tag{1.154}$$

In particular, the function $\xi(t)$ can be sought in the form

$$\xi(t) = \begin{cases} a(1-\cos t), & \text{for } 0 < t < \theta_1 \\ -t + 2\theta_2 - \beta, & \text{for } \theta_1 \leq t < \theta_2 \\ t - \beta, & \text{for } \theta_2 \leq t < t_1(\beta) \end{cases}, \tag{1.155}$$

where

$$\theta_2 = \frac{\theta_1 + a(1-\cos\theta_1) + \beta}{2} \tag{1.156}$$

and the parameter θ_1 can be found from Eq. (1.154). The construction of the function $\xi(t)$ of Eq. (1.155) is illustrated in Fig. 1.13. The plot of this function is shown by a thick line. Then the optimal control, according to Eq. (1.120), takes the form

$$u_0(t) = \begin{cases} a\sin t, & \text{for } 0 \leq t < \theta_1 \\ -1, & \text{for } \theta_1 \leq t < \theta_2 \\ 1, & \text{for } \theta_2 \leq t < t_3(\beta) \\ v(t), & \text{for } t \geq t_3(\beta) \end{cases}. \tag{1.157}$$

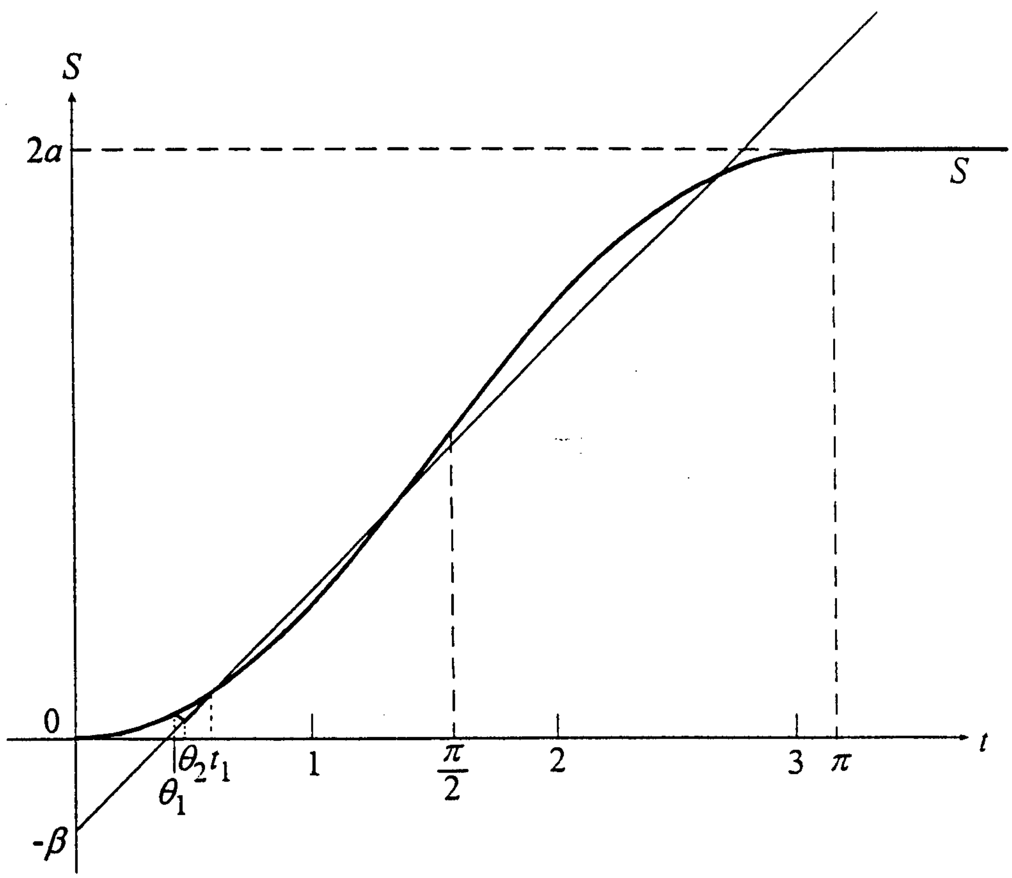


Figure 1-13. The function $\xi(t)$.

The minimum of the peak displacement of the body being isolated is calculated according to Eq. (1.121) and is given by

$$J_1(u_0) = -A_2^{\tilde{U}}(\beta)/2. \tag{1.158}$$

Case $a \le a_*$. In this case, the construction of the optimal control completely coincides with that for the case $\Omega_2 < \Omega_1$ for $a > a_*$.

1.2.2.14 The Structure of the Optimal Control. Gather together the formulas of Eqs. (1.141), (1.152), and (1.157) for the optimal control

$$u_0(t) = \begin{cases} a \sin t, & \text{for } 0 \le t < \vartheta_1 \\ -1, & \text{for } \vartheta_1 \le t < \vartheta_2 \\ 1, & \text{for } \vartheta_2 \le t < \vartheta_3 \\ v(t), & \text{for } t \ge \vartheta_3 \end{cases}. \tag{1.159}$$

The switching times $\vartheta_1, \vartheta_2,$ and ϑ_3 depend on the dimensionless amplitude a of the disturbance. Specifically, for the control of Eq. (1.141), we have $\vartheta_1 = 0, \vartheta_2 = 0, \vartheta_3 = 2a$; for the control of Eq. (1.152), $\vartheta_1 = 0, \vartheta_2 = \delta/2, \vartheta_3 = t_3(\delta)$; for the control of Eq. (1.157), $\vartheta_1 = \theta_1, \vartheta_2 = \theta_2, \vartheta_3 = t_3(\beta)$.

1.2.2.15 Numerical Results. Some numerical results are presented in Figs. 1.14 to 1.20. Figure 1.14 shows the optimal value of the performance index J_1 of Eq. (1.124) versus the disturbance amplitude $a \ge 1$. Figures 1.15 to 1.17 show the switching times $\vartheta_1, \vartheta_2,$ and ϑ_3 of the optimal control of Eq. (1.159) versus a.

It is seen from Figs. 1.15 to 1.17 that all the switching times approach $\pi/2$ as $a \to 1$. In this case, as follows from Eq. (1.159), the optimal control becomes identically equal to the disturbance function. As a increases above the value $a = 1$, the functions $\vartheta_1(a)$ and $\vartheta_2(a)$ monotonically decrease, whereas $\vartheta_3(a)$ monotonically increases. For $1 < a < 1.25$, the optimal control has the form of Eq. (1.157). The control of such a kind is shown in Fig. 1.18. At $a \approx 1.25$, the switching time $\vartheta_1(a)$ becomes zero, and the control of Eq. (1.157) transforms to the control of Eq. (1.152). See Fig. 1.19. For $1.25 < a < 1.65$, the optimal control has the form of Eq. (1.152). At $a \approx 1.65$, the switching time ϑ_2 in Eq. (1.159) vanishes, and the control acquires the form of Eq. (1.141), as shown in Fig. 1.20. This form of the optimal control is preserved for all $a > 1.65$.

It is evident from Fig. 1.14 that the optimal value of the performance index grows without limit as a increases. Note that for $a > 1.65$ the optimal value of the performance index is given by the closed-form relation of Eq. (1.142)

$$J_1 = a(2a - \pi). \tag{1.160}$$

1.2.2.16 Conclusion. Above, we have outlined briefly the most widespread analytical methods of solving the problem of limiting isolation capabilities that can be applied to systems with isolators. For more detail, we recommend the monographs by Sevin and Pilkey (1971) and Kolovskii (1976), as well as such papers as Guretskii (1965a,b,c, 1969a), Viktorov and Larin (1969), Saranchuk and Troitskii (1969), Wang and Pilkey (1975), and others dealing with numerous analytical and numerical approaches to the solution of the problem of limiting isolation capabilities.

1.2.3 Synthesis of Optimal Feedback Characteristics.

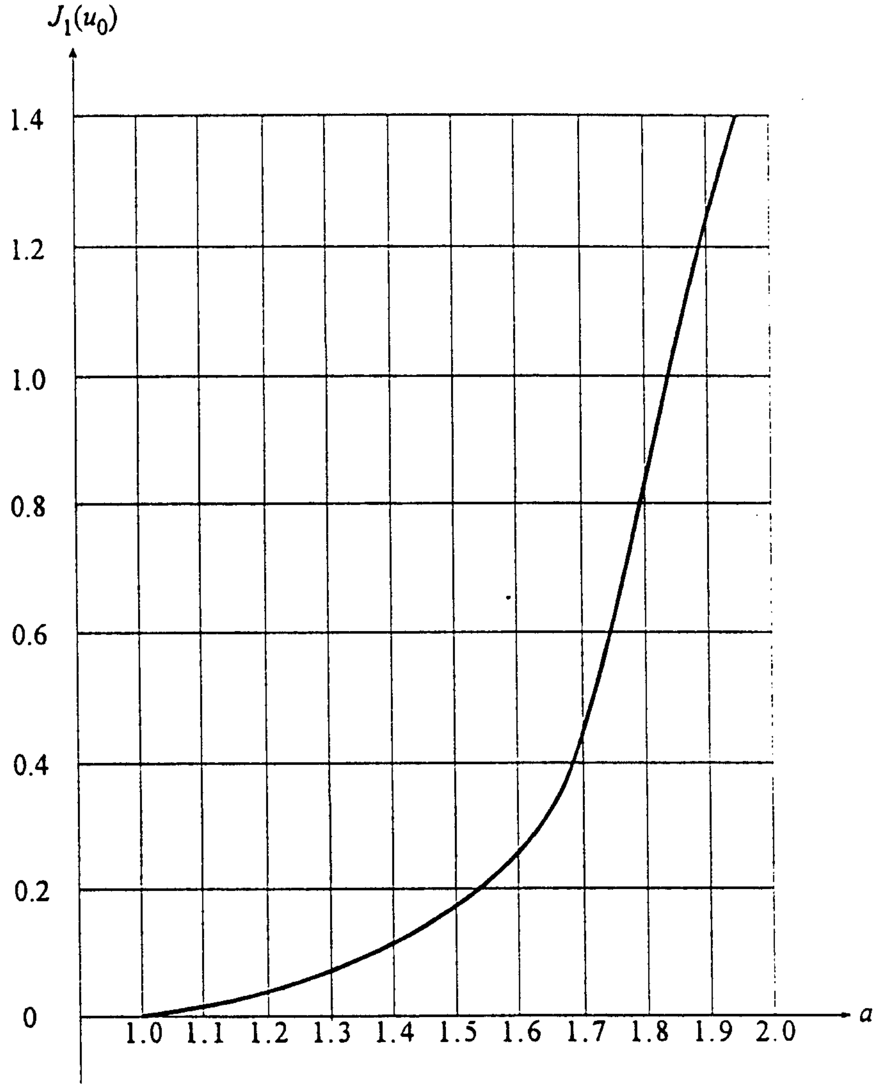


Figure 1-14. Minimum peak displacement versus the pulse amplitude.

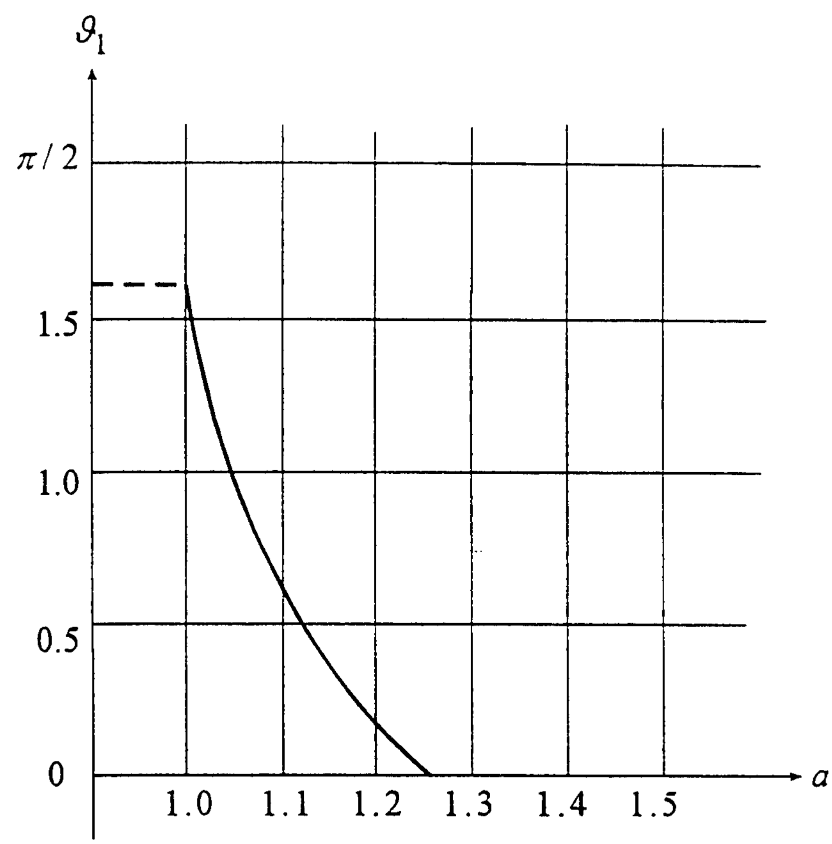


Figure 1-15. The switching time ϑ_1 versus the pulse amplitude.

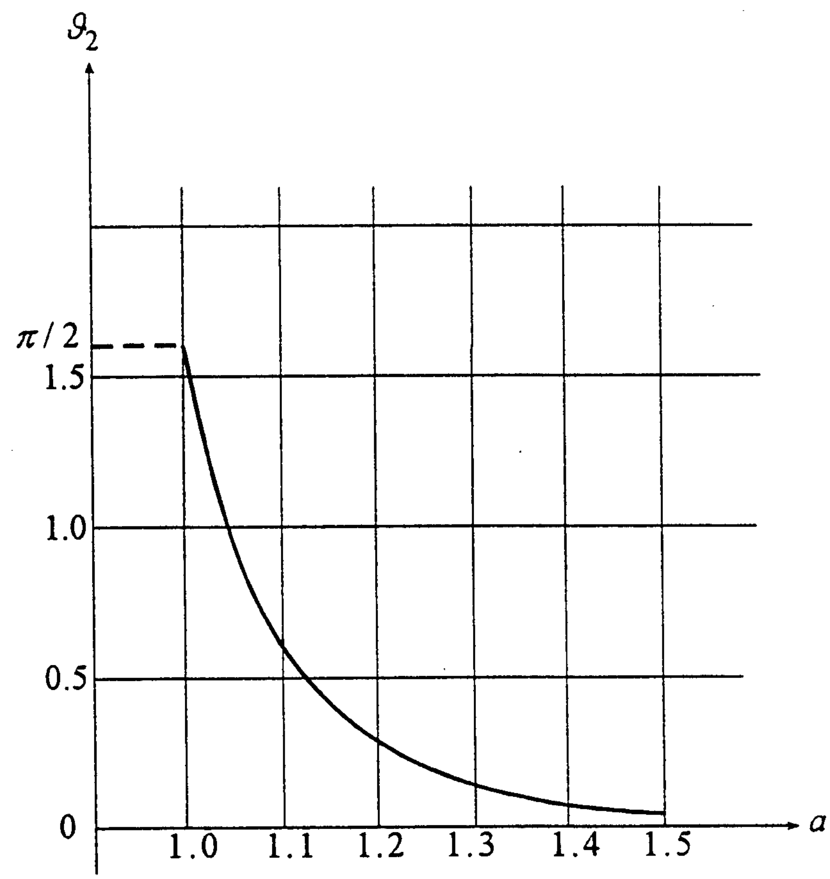


Figure 1-16. The switching time ϑ_2 versus the pulse amplitude.

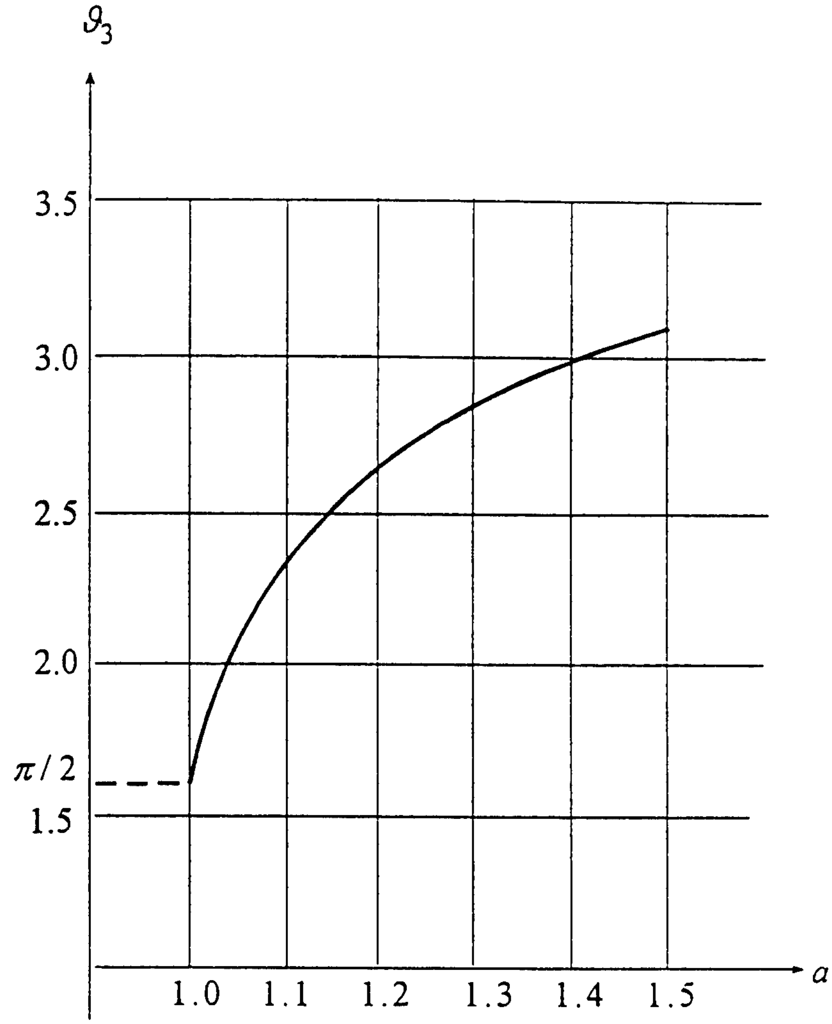


Figure 1-17. The switching time ϑ_3 versus the pulse amplitude.

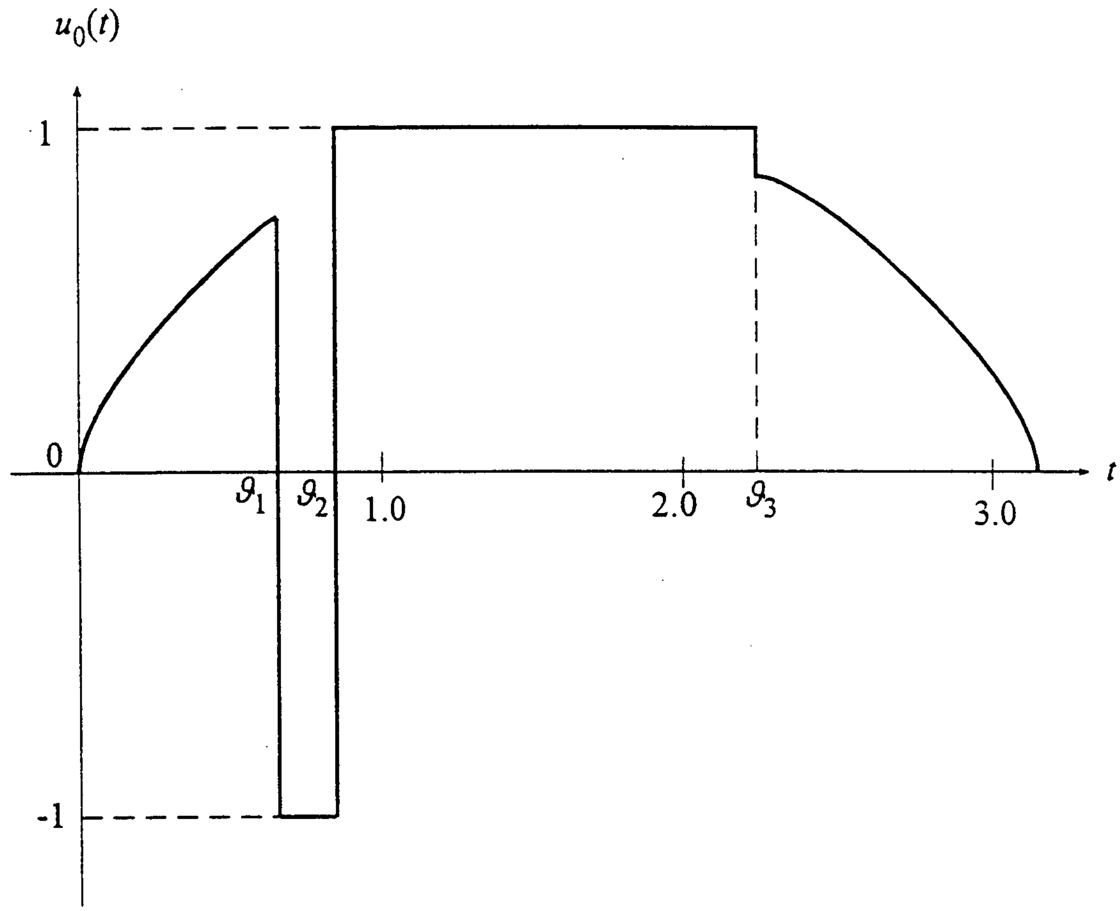


Figure 1-18. Time history of the optimal control for 1<a<1.25.

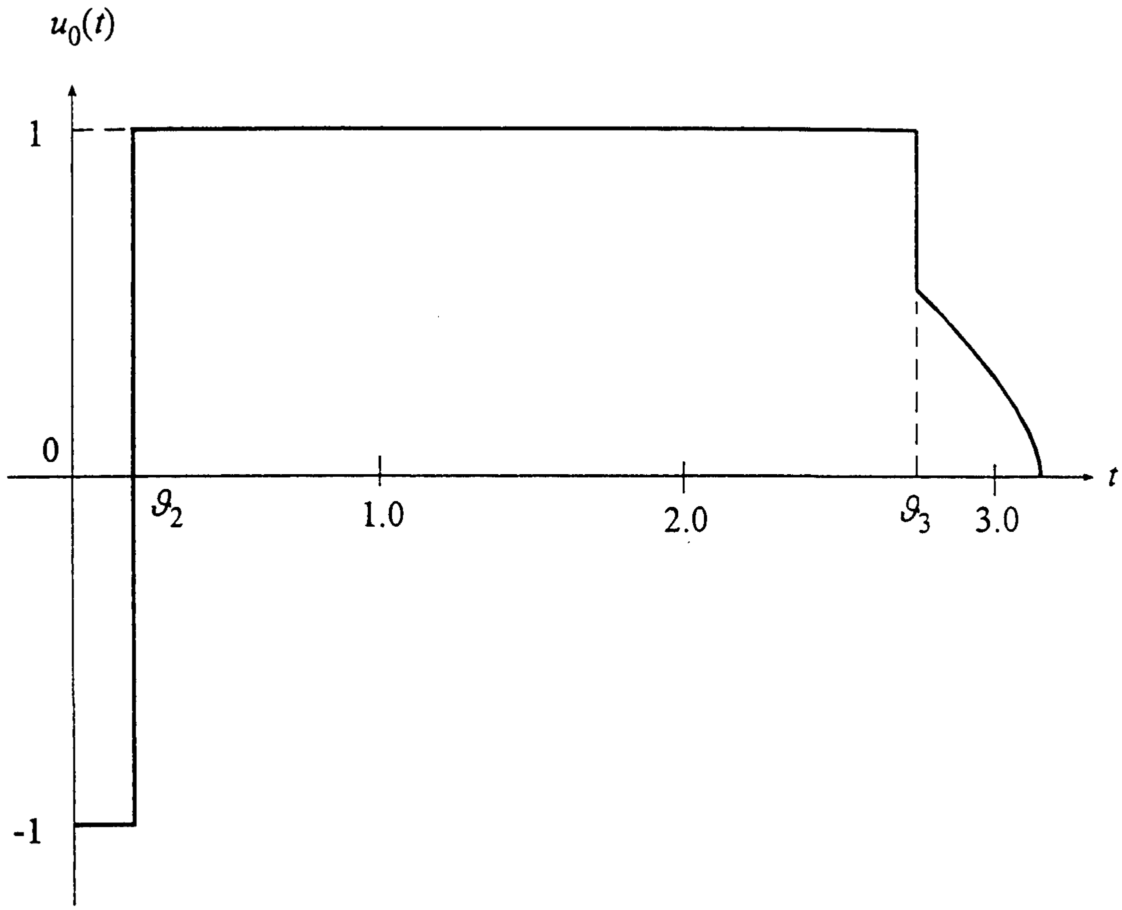


Figure 1-19. Time history of the optimal control for $1.25<a<1.65$.

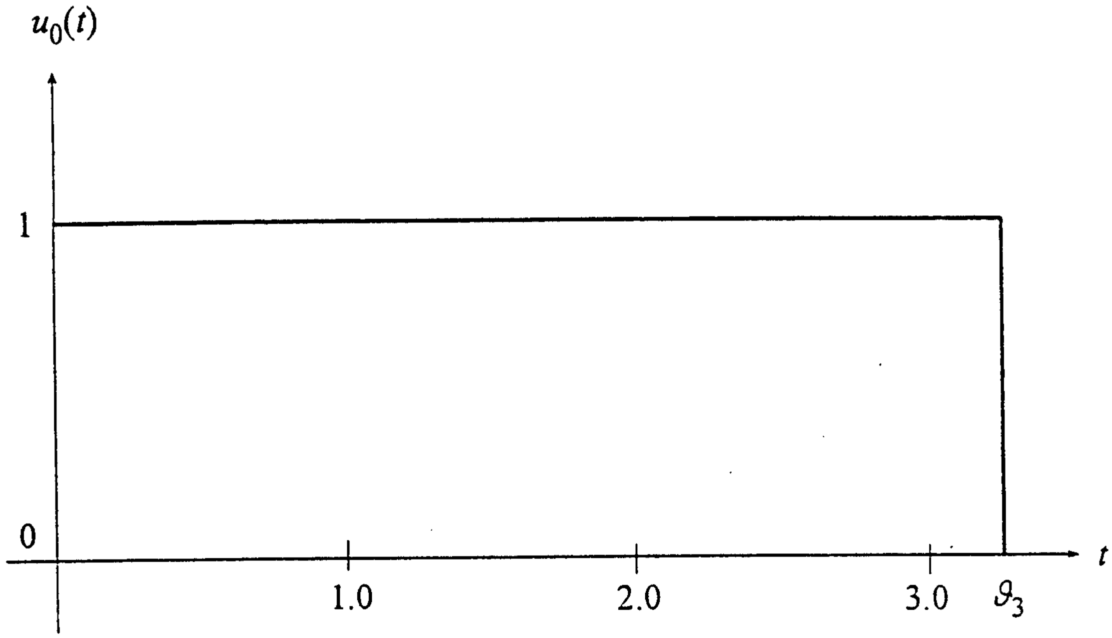


Figure 1-20. Time history of the optimal control for a>1.65.

1.2.3.1 Necessity of the Feedback Control. The solution of the problem of limiting isolation capabilities yields the optimal isolator characteristic in the form of an open-loop control which is a function of time alone. Such a characteristic provides the minimum value for the optimization criterion with all imposed constraints being satisfied only in case the motion of the system follows exactly the single state trajectory (nominal trajectory) corresponding to the specified initial conditions for which the optimal control was calculated. This open-loop control in no way reacts to deviations of the state variables from their nominal values and in the general case does not ensure stability of the nominal motion. Thus, in practice, an actual system may have unacceptably large values of the optimization criterion, e.g., the relative displacement of a body being isolated.

Consider a simple example. Let the relative motion of a body being isolated be described by the equation $\ddot{x} + u = v(t)$ (Eq. 1.74), while the external disturbance $v(t)$ satisfies the conditions of Eq. (1.95) with $\tau_1 = 0$. Then, as shown in the previous section, the open-loop control $u_0(t)$ given by Eq. (1.96) provides the minimum of the maximum absolute value of the displacement of the body being isolated, $\min_u \max_t |x_u(t)| = |x_{u_0}(t_*)|$, provided the absolute value of the force acting on it does not exceed the prescribed value u_* and at the initial instant $t = 0$ the body is in the state $x(0) = 0, \dot{x}(0) = 0$.

The general solution of Eq.(1.74) is

$$x(t) = x_0 + \dot{x}_0 t + \int_0^t (t - \tau)[v(\tau) - u(\tau)]d\tau, \tag{1.161}$$

where x_0 and $\dot{x}_0$ are the values of the variable x and its derivative with respect to time at the instant $t = 0$. According to Eq. (1.161), the nominal state trajectory corresponding to the open-loop control $u_0(t)$ defined by Eq. (1.96) with zero initial conditions is expressed as

$$\begin{aligned} x(t) &= \textstyle\int_0^t (t - \tau)[v(\tau) - u_*]d\tau, \quad 0 \le t < t_*; \\ x(t) \equiv x(t_*) &= \textstyle\int_0^{t_*} (t_* - \tau)[v(\tau) - u_*]d\tau, \quad t \ge t_*; \end{aligned} \tag{1.162}$$

$$\begin{aligned} \dot{x}(t) &= \textstyle\int_0^t [v(\tau) - u_*]d\tau, \quad 0 \le t < t_*; \\ \dot{x}(t) &\equiv 0, \quad t \ge t_*. \end{aligned}$$

According to Eq. (1.162), the body being isolated comes to rest at the instant $t = t_*$ at the position $x = x(t_*)$ corresponding to the maximum displacement and remains in this position for unlimited time. However, if the initial conditions are nonzero, then, according to Eq. (1.161), we have $x = x_0 + x(t_*) + \dot{x}_0 t$ for $t \ge t_*$, provided that the control $u = u_0(t)$ is applied. Hence, the maximum displacement becomes arbitrarily large even if $\dot{x}_0$ differs from zero by a small amount.

It becomes apparent that in practical isolation system design, it is usually essential to design the isolator characteristic as a feedback control, which depends on the state variables and, for active isolation systems, also on time. As a rule, when synthesizing a feedback isolator characteristic, it is required to provide the stability of the nominal motion so as to mitigate unavoidable small deviations of the state variables from their nominal values and thus prevent performance indices from substantially exceeding the precalculated values.

1.2.3.2 Problem of Synthesis of Optimal Feedback Isolator Characteristics. Consider the problem of synthesis of the optimal feedback isolator characteristic in the case where the external disturbance is specified and the motion of the system with isolators is governed by the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$, $t_0 \leq t \leq T$, subject to the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$. (See Eq. 1.72.)

1.2.3.3 Problem 1.7. Find from among a specified set U of functions of state variables and, possibly, time the optimal characteristic $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}, t)$ such that

$$J_1(\mathbf{u}_0) = \min_{u \in U}\{J_1(\mathbf{u}) \mid J_i(\mathbf{u}) \leq D_i, \quad i = 2, \ldots, N\}. \tag{1.163}$$

Problem 1.7, as well as Problem 1.4 of the limiting isolation capabilities, is a particular case of the general optimal isolation problem formulated as Problem 1.1.

The statement of Problem 1.7 resembles the mathematical problem of synthesizing optimal feedback control as it is considered in the theory of optimal control (Pontryagin, et al., 1962; Lee and Markus, 1967; Boltyanskii, 1968; Moiseev, 1975;). However, there is an essential difference between these problems. In the mathematical theory of optimal control, the synthesis of the optimal feedback control is defined as the formation of a function $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}, t)$ which possesses the following two properties. First, in the general case, this function depends on all dynamic variables, i.e., on both the state variables and time. Second, the control $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}, t)$ provides the optimality for any state trajectory of the system governed by Eq. (1.72) with $\mathbf{u}_0(\mathbf{x}, t)$ being substituted for $\mathbf{u}$, under arbitrary initial conditions. The realization $\widetilde{\mathbf{u}}(t) = \mathbf{u}_0(\mathbf{x}_{u_0}(t), t)$ of the control $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}, t)$ for each particular trajectory of the system is the optimal open-loop control providing the absolute minimum for the optimization criterion when moving along this trajectory. Such a problem has a solution if the admissible class of functions $\mathbf{u}(\mathbf{x}, t)$, among which the optimal feedback control is sought, is sufficiently wide. In the theory of optimal control, this class is not narrower than the set of piecewise continuous functions of state variables and time.

1.2.3.4 Dynamic Programming and the Principle of Optimality A rather general approach to the synthesis of the optimal feedback control is given by Bellman's method of dynamic programming (Bellman, 1957b, Bellman and Dreyfus, 1962). This method is developed in depth for optimal control problems with the so-called *additive functionals* of the form of Eq. (1.81). For such functionals the *principle of optimality* is valid. According to this principle, any segment of the optimal trajectory is itself an optimal trajectory connecting the endpoints of the indicated segment. The rigorous mathematical formulation of the principle of optimality is given next.

Let $\mathbf{u}_0(t)$ and $\mathbf{x}_{u_0}(t)$ be the optimal open-loop control and the corresponding state trajectory of the system of Eq. (1.72) that are defined over the time interval $[t_0, T]$, satisfy specified initial conditions, and minimize the functional of Eq. (1.81). Then for any $t_* \in [t_0, T]$, the control $\mathbf{u}_0(t)$ restricted to the time interval $[t_*, T]$ minimizes the functional

$$\int_{t_*}^{T} \Phi_1(\mathbf{x}_u(t), \mathbf{u}(t), t)dt + \Phi_2(\mathbf{x}_u(T), T)$$

with respect to trajectories of the system of Eq. (1.72) passing through the point $\mathbf{x}_{u_0}(t_*)$ at the instant $t = t_*$. The corresponding optimal trajectory coincides with $\mathbf{x}_{u_0}(t)$ in the interval $[t_*, T]$.

To find the optimal feedback control by using the dynamic programming method, an initial-value problem for a nonlinear partial differential equation of a special kind (Bellman's equation) is solved. For details, see Bellman (1957b, Bellman and Dreyfus, 1962), Boltyanskii (1968), or Roitenberg (1978). There are references extending the principle of optimality and the method of dynamic programming to problems where the optimization criterion is the maximum over time of a function of state variables and time (Bellman, 1957a).

The application of dynamic programming to the calculation of optimal isolator characteristics is discussed in Sevin and Pilkey (1971).

The prospects for the use of dynamic programming for the synthesis of optimal isolator characteristics are rather limited, mostly because of the large number of computer operations and the sizeable memory necessary for its implementation. The so-called "curse of dimensionality" is caused by memory requirements that grow exponentially with the increase of the system dimensionality.

For analytical (closed-form) synthesis of the optimal feedback control, dynamic programming is useful for special cases only. For this reason, in this book, we do not discuss the method of dynamic programming in detail. Interested readers can consult such literature as Bellman (1957b), Bellman and Dreyfus (1962), and Moiseev (1975).

In many practical cases of designing isolation systems, the set of admissible characteristics is more restricted than the set of piecewise continuous functions of state variables and time. For example, often it is required to provide the best isolation quality using only passive isolators, whose characteristics do not depend on time explicitly, or isolators of a specified structure with the characteristics containing numerical parameters to be determined by means of optimization. If the set of admissible controls is restricted as indicated, then it is generally impossible to synthesize the unified characteristic $\mathbf{u}_0(\mathbf{x}, t)$ which provides the optimal isolation for any initial conditions. Problem 1.7 pursues a more modest aim of determining the isolator characteristic $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}, t)$ that minimizes the optimization criterion only for specified initial conditions. Note that this characteristic does not necessarily provide the limiting possible isolation quality if the set U of admissible characteristics is not sufficiently broad. If the set U of admissible isolator characteristics contains a function $\mathbf{u}(\mathbf{x}, t)$ ensuring the optimal isolation quality under any initial conditions, then this function is the solution to Problem 1.7. In our opinion, in the practice of isolation system design, one can often use for the synthesis of the optimal characteristic the approach associated with the replacement of the initial set of admissible characteristics by an appropriate parametric family of functions and subsequent optimization of these parameters. Such an approach is discussed in detail in Section 1.3.

1.2.4 Multicriteria Optimization.

As mentioned earlier, isolation system design problems tend to be multicriteria in nature, because the system to be designed may need to meet multiple requirements. Then the designer has to take into account several performance indices of the form of Eqs. (1.56) or (1.57). One of the possible approaches to the rational calculation of isolator characteristics when several performance indices are involved is to solve Problem 1.1 that was formulated in Section 1.2.1. This rather common approach involves minimizing one of the performance criteria while the others are subject to

appropriate constraints. This can lead to favorable results if the most critical performance criterion can be identified and then minimized, while for the other performance criteria, estimates of the maximum admissible values assuring reliable functioning of the system are available. However, this situation does not always occur, and one has to use alternative approaches for the isolator design. Two such alternatives are outlined below.

1.2.4.1 Minimization of a Weighted Criterion. The first method involves minimizing with respect to $\mathbf{u} \in U$ the weighted performance criterion specified by the linear combination

$$\widetilde{J}(\mathbf{u}) = \sum_{i=1}^{N} a_i \widetilde{J}_i(\mathbf{u}), \quad a_i \geq 0, \quad \sum_{i=1}^{N} a_i = 1 \tag{1.164}$$

of the functionals of Eq. (1.58). The coefficients a_i, $i = 1, \ldots, N$, are weighting factors that reflect the relative influence of particular functionals $\widetilde{J}_i(\mathbf{u})$ on the general performance of the system under design. The factors a_i should be chosen carefully, taking into consideration the knowledge of the relative importance of the functionals. Such a method has been used, for example, in Karnopp and Trikha (1969).

1.2.4.2 Pareto-Optimal Solutions. The other approach involves the determination of Pareto-optimal (Karlin, 1959) characteristics and/or parameters of the isolation system. We begin with the mathematical definition of Pareto-optimality.

1.2.4.3 Definition 1.1. Let N functionals $\widetilde{J}_i(\mathbf{u}), i = 1, \ldots, N$, be defined over the set U $(\mathbf{u} \in U)$. The element $\mathbf{u}_0 \in U$ is called Pareto-optimal if for any element $\mathbf{u} \in U$ different from $\mathbf{u}_0$, $\mathbf{u} \neq \mathbf{u}_0$, there exists the index $k \in \{i = 1, \ldots, N\}$ such that $\widetilde{J}_k(\mathbf{u}) > \widetilde{J}_k(\mathbf{u}_0)$ or the inequality $\widetilde{J}_j(\mathbf{u}) \geq \widetilde{J}_j(\mathbf{u}_0)$ holds for any $j \in \{i = 1, \ldots, N\}$.

The set Π of all Pareto-optimal elements $\mathbf{u}_0 \in U$ is called the Pareto-optimal set for the system of functionals $\widetilde{J}_i(\mathbf{u}), (i = 1, \ldots, N)$. Obviously, $\Pi \subset U$.

Definition 1.1 implies nothing about the mathematical nature and interpretation of functionals $\widetilde{J}_i(\mathbf{u}), i = 1, \ldots, N$, and the set U. These depend on the nature (mechanical, economical, etc.) of the system under investigation, for which the Pareto-optimal state or operation mode is sought. In our case, U is the set of admissible characteristics of isolators, while $\widetilde{J}_i(\mathbf{u}), i = 1, \ldots, N$, are performance criteria of isolation. More precisely, the quantities $\widetilde{J}_i(\mathbf{u})$ are the values of the performance criteria corresponding to characteristics $\mathbf{u}$ under the least favorable external disturbance (Eq. 1.58).

Definition 1.1 implies that there is no argument $\mathbf{u} \in U$ that can be chosen to reduce simultaneously all the functionals $\widetilde{J}_i(\mathbf{u}), i = 1, \ldots, N$, compared with their values at any point of the Pareto-optimal set Π. Thus, optimization in the sense of Pareto permits one to find out to what extent the simultaneous reduction of all $\widetilde{J}_i(\mathbf{u}), i = 1, \ldots, N$, is possible. Investigation of such a possibility seems to be rather important when designing isolation systems, especially in the cases where it is difficult to assess to what extent a particular mechanical characteristic, which is the basis of the corresponding functional $\widetilde{J}_i(\mathbf{u})$, influences the general behavior of the system. In these cases, it is advisable to construct the Pareto-optimal set for admissible characteristics of isolators

at the initial stage of designing the isolation system, and then choose the final characteristic from among Pareto-optimal ones, relying on additional requirements to the system to be designed.

The approach to the calculation of characteristics for isolators based on the construction of the Pareto-optimal set was used in Bolychevtsev and Lavrovskii (1977) and Rao and Hati (1980).

1.2.4.4 Connection Between the Optimization of a Weighted Criterion and

Pareto Optimization: Between the two approaches described above, there exists the connection established by

1.2.4.5 Proposition 1.2. The element $\mathbf{u}_0$ of the set U minimizing the functional $\widetilde{J}(\mathbf{u})$ of Eq. (1.164) with all $a_i > 0$, $\quad i = 1, \ldots, N$, is Pareto-optimal for the system of functionals $\widetilde{J}_i(\mathbf{u})$, $i = 1, \ldots, N$.

In other words, the fact that at some $\mathbf{u}_0 \in U$ the functional $\widetilde{J}(\mathbf{u})$ with all $a_i > 0$ assumes its minimum value is a sufficient condition for Pareto-optimality of the element $\mathbf{u}_0 \in U$.

1.2.4.6 Proof. If $\mathbf{u}_0 \notin \Pi$, then, according to Definition 1.1, there exists an element $\hat{\mathbf{u}} \in U$ such that $\widetilde{J}_i(\hat{\mathbf{u}}) \leq \widetilde{J}_i(\mathbf{u}_0)$ for all $i = 1, \ldots, N$ and, for at least one $k \in \{i = 1, \ldots, N\}$, $\widetilde{J}_k(\hat{\mathbf{u}}) < \widetilde{J}_k(\mathbf{u}_0)$. Hence, $\widetilde{J}(\hat{\mathbf{u}}) < \widetilde{J}(\mathbf{u}_0)$ if all $a_i > 0$ in Eq. (1.164). Therefore, if the element $\mathbf{u}_0$ is not Pareto-optimal, it cannot minimize the functional $\widetilde{J}(\mathbf{u})$. Thus, any isolator characteristic that minimizes the weighted performance index of Eq. (1.164), where $a_i > 0$ for $i = 1, \ldots, N$, is Pareto-optimal.

The minimality of the functional $\widetilde{J}(\mathbf{u})$ of the form of Eq. (1.164) at some $\mathbf{u}_0 \in U$, being a sufficient condition, is not a necessary condition of Pareto-optimality of the element $\mathbf{u}_0$. This can be shown for the simple example with $N = 2$. Let $\widetilde{J}_1(u)$ and $\widetilde{J}_2(u)$ be functions of a single variable u defined over the real axis ($U = (-\infty, +\infty)$) as follows:

$$\widetilde{J}_1(u) = \begin{cases} 2u, & u < 0 \\ u, & u \geq 0 \end{cases}, \tag{1.165}$$

$$\widetilde{J}_2(u) = \begin{cases} -3u, & u < 0 \\ -2u, & u \geq 0 \end{cases}. \tag{1.166}$$

Graphs of functions $\widetilde{J}_1(u)$ and $\widetilde{J}_2(u)$ are shown in Fig. 1.21 by solid and dashed lines, respectively. The weighted functional $\widetilde{J}(u)$ in the case in question is represented by

$$\widetilde{J}(u) = a\widetilde{J}_1(u) + (1-a)\widetilde{J}_2(u) = \begin{cases} (5a-3)u, & u < 0 \\ (3a-2)u, & u \geq 0 \end{cases}, \tag{1.167}$$

It is evident that the point $u = 0$ is Pareto-optimal for the system of functions $\widetilde{J}_1(u)$ and $\widetilde{J}_2(u)$. However, from Eq. (1.167), the value $u = 0$ cannot minimize the function of Eq. (1.167) for any admissible value of the parameter a. Indeed, for the function $\widetilde{J}(u)$ to assume its minimum value at $u = 0$, i.e. for the inequality $\widetilde{J}(u) \geq \widetilde{J}(0)$ to hold for all u, the parameter a must simultaneously

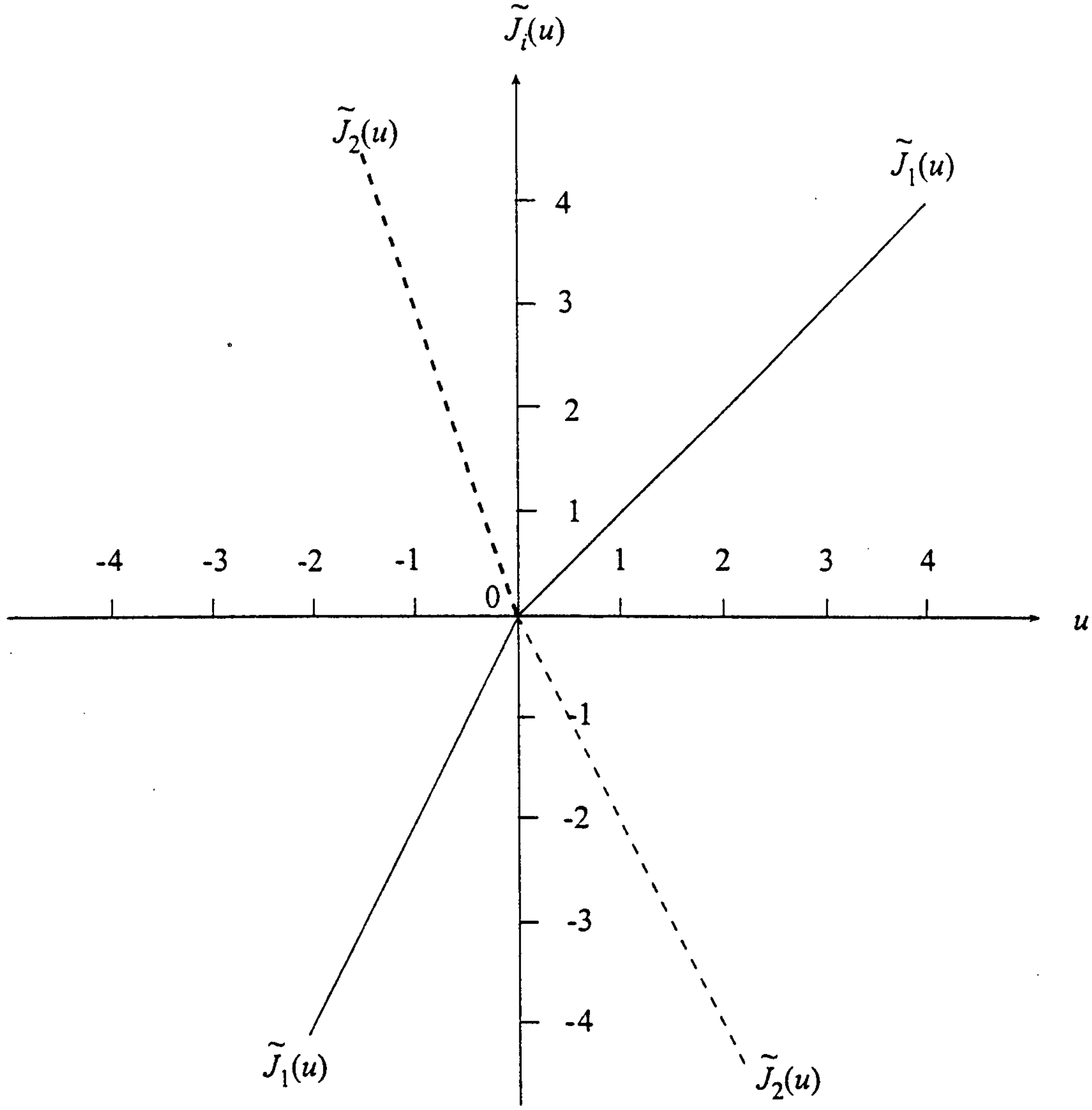


Figure 1-21. Functions $\tilde{J}_1(u)$ and $\tilde{J}_2(u)$ for which the coordinate origin is the Pareto optimal point that cannot be found by minimization of the linear combination of these functions.

satisfy the inequalities $a \leq 3/5$ (for $u < 0$) and $a \geq 2/3$ (for $u > 0$), and these inequalities are inconsistent because $2/3 > 3/5$.

One should take this fact into account when using methods based on Proposition 1.2 for finding Pareto-optimal points. If we construct the set of controls u_a that minimize the functional of Eq. (1.164) for all $a_i > 0$ satisfying the relation $\sum_{i=1}^{N} a_i = 1$, we obtain, in the general case, only a subset of the Pareto-optimal set. Such a method has been used, for example, in Rao and Hati (1980).

1.2.4.7 Pareto-Optimal Set for Design Variables of a Linear Isolator. Let us construct the Pareto-optimal set for design variables of the linear spring-and-damper isolator. Consider a body connected with a rectilinearly moving base by means of an isolator and able to move relative to the base in the direction of the latter's motion. See Fig. 1.1. Assume that at the instant $t = 0$ the body being isolated is in the state corresponding to $x(0) = 0$ and $\dot{x}(0) = 0$, while the base is subject to the kinematical shock $\ddot{y} = -\beta\delta(t)$. Here, x is the coordinate of the body being isolated related to the base-connected reference frame, $x = 0$ corresponding to the undeformed spring; y is the coordinate of the base with respect to the inertial reference frame; $\delta(t)$ is Dirac's delta-function; and β is the constant characterizing the shock intensity and equal to the jump-like change of the velocity of the base resulting from the shock. In this case, the motion of the body being isolated with respect to the base is described by the differential equation of Eq. (1.2) with zero initial conditions and $g(x, \dot{x}, t) = -c\dot{x} - kx$, that is

$$m\ddot{x} + c\dot{x} + kx = m\beta\delta(t), \quad x(0) = \dot{x}(0) = 0, \quad m > 0, c \geq 0, k \geq 0, \tag{1.168}$$

where m is the mass of the body being isolated and k and c are the stiffness and damping coefficients of the isolator, respectively.

Without loss of generality, assume $m = 1$ and $\beta = 1$. This corresponds to using in Eq. (1.168) the dimensionless (primed) variables

$$\begin{array}{ll} x' = \frac{x}{D} sign\beta, & t' = \frac{|\beta|}{D} t, \\ c' = \frac{D}{|\beta| m} c, & k' = \frac{D^2}{\beta^2 m} k. \end{array} \tag{1.169}$$

The quantity D denotes some positive constant of the dimension of length that is used as a standard for measuring lengths when establishing the variables of Eq. (1.169).

As mentioned previously, the maximum displacement of the body being isolated relative to the base,

$$J_1(c, k) = \max_{t \in [0,\infty)} |x(t, c, k)|, \tag{1.170}$$

and the maximum absolute value of the force transmitted to the body being isolated by the isolator,

$$J_2(c, k) = \max_{t \in [0,\infty)} |c\dot{x}(t, c, k) + kx(t, c, k)|. \tag{1.171}$$

are common performance criteria of isolation in the case of shock disturbances.

Construct the Pareto-optimal set in the plane of the isolator parameters c and k which are the design variables. In our case, the set U is the first quadrant ($c \geq 0, k \geq 0$) of the parameter plane. Solving the initial-value problem of Eq. (1.168) and calculating the maxima in Eqs. (1.170) and (1.171) lead to the expressions

$$J_1(c,k) = \begin{cases} \frac{1}{\sqrt{k}} \left[\frac{c-\sqrt{c^2-4k}}{c+\sqrt{c^2-4k}}\right]^{\frac{c}{2\sqrt{c^2-4k}}}, & c^2-4k>0 \\ \frac{2}{c}\exp(-1), & c^2-4k=0 \\ \frac{1}{\sqrt{k}}\exp\left[-\frac{c}{\sqrt{4k-c^2}}\tan^{-1}\frac{\sqrt{4k-c^2}}{c}\right], & c^2-4k<0 \end{cases} \tag{1.172}$$

$$J_2(c,k) = \begin{cases} c, & c^2-k \geq 0 \\ \sqrt{k}\exp\left[-\frac{c}{\sqrt{4k-c^2}}\tan^{-1}\frac{(k-c^2)\sqrt{4k-c^2}}{3ck-c^3}\right], & c^2-k<0 \end{cases} \tag{1.173}$$

for $J_1(c,k)$ and $J_2(c,k)$.

Level curves of the functions of Eqs. (1.172) and (1.173) are shown in Fig. 1.22 by solid and dashed curves, respectively. Level curves of the function of Eq. (1.172) are convex, while those for the function of Eq. (1.173) are concave.

It can be shown that the function of Eq. (1.172) monotonically decreases, while that of Eq. (1.173) monotonically increases when moving away from the origin ($c = 0, k = 0$) along any ray passing through the origin and lying in the first quadrant ($c \geq 0, k \geq 0$).

Consider the domain $U = \{c, k : c \geq 0, k \geq 0\}$ and an arbitrary point ψ, the coordinates of which are denoted by $c = c_*$ and $k = k_*$ and consider the sets

$$\Omega_{J_i}(\psi) = \{(c,k) \in U : J_i(c,k) \leq J_i(c_*, k_*)\}, \quad i = 1, 2. \tag{1.174}$$

These sets are easy to construct based on the analysis of the level curves of performance criteria $J_1(c,k)$ and $J_2(c,k)$ depicted in Fig. 1.22. By virtue of the aforementioned properties of the functions of Eqs. (1.172) and (1.173), the set $\Omega_{J_1}(\psi)$ consists of the points lying above the level curve $J_1(c,k) = J_1(c_*, k_*)$ of the function $J_1(c,k)$ or on the line itself, while the set $\Omega_{J_2}(\psi)$ consists of the points lying below the level curve $J_2(c,k) = J_2(c_*, k_*)$ of the function $J_2(c,k)$ or on the line itself.

The definition of Pareto-optimality and the definition of the sets $\Omega_{J_1}(\psi)$ and $\Omega_{J_2}(\psi)$ imply that the point ψ is Pareto-optimal if and only if the intersection of the sets $\Omega_{J_1}(\psi)$ and $\Omega_{J_2}(\psi)$ coincides with the intersection of the level curves of the functions $J_1(c,k)$ and $J_2(c,k)$ passing through the point ψ, i.e.,

$$\Omega_{J_1(\psi)} \cap \Omega_{J_2(\psi)} = \{c, k : J_1(c,k) = J_1(c_*, k_*)\} \cap \{c, k : J_2(c,k) = J_2(c_*, k_*)\} \tag{1.175}$$

Accordingly, to construct the Pareto-optimal set it is sufficient to construct the sets $\Omega_{J_1}(\psi)$ and $\Omega_{J_2}(\psi)$ for all $\psi \in U$ and check for the satisfaction of the relation of Eq. (1.175). The set of all admissible points for which Eq. (1.175) holds is the desired Pareto-optimal set.

The analysis of the mutual arrangement of the level curves of the functions $J_1(c, k)$ and $J_2(c, k)$ on the ck-plane (see Fig. 1.22) shows that for each level curve of the function $J_1(c, k)$ or $J_2(c, k)$, there exists one and only one point ψ for which Eq. (1.175) is satisfied. This point is the tangent point of the corresponding level curves.

Consider an arbitrary level curve γ_D of the function $J_1(c, k)$ corresponding to the value D of this function, i.e.,

$$\gamma_D = \{(c, k) : c \geq 0, k \geq 0, \quad J_1(c, k) = D\}. \tag{1.176}$$

It follows from the aforementioned properties of the functions $J_1(c, k)$ and $J_2(c, k)$ that the tangent point of the curve γ_D and the corresponding level curve of the function $J_2(c, k)$ is the point of the minimum of the performance criterion $J_2(c, k)$ over the curve γ_D. Thus, for determining the contact point it is sufficient to find parameters c and k minimizing the function $J_2(c, k)$ over the curve γ_D. This problem is reduced to searching for the minimum of a single-variable function that can be implemented easily with a computer.

Denote the coordinates of the desired point by c_D and k_D. If D is chosen to be the constant of dimension of length in the formulas of Eq. (1.169) for the dimensionless variables, then the following relations are implied:

$$c_D = \frac{m\,|\beta|}{D} c_1, \quad k_D = \frac{m\beta^2}{D^2} k_1, \tag{1.177}$$

where c_1 and k_1 are the coordinates of the point of the minimum of the function $J_2(c, k)$ on the curve

$$\gamma_1 = \{(c, k) : c \geq 0, k \geq 0, \quad J_1(c, k) = 1\}, \tag{1.178}$$

when $m = 1$ and $\beta = 1$. Calculations yield

$$c_1 = 0.485, \quad k_1 = 0.361. \tag{1.179}$$

It follows from Eq. (1.177) that the desired Pareto-optimal set is the curve (parabola), for which the parametric representation is given by Eq. (1.177), with D being the parameter of the curve. Eliminating D from Eq. (1.177) and taking into account Eq. (1.179) we can represent the Pareto-optimal set Π as

$$\Pi = \{c, k : c \geq 0, k \geq 0, \quad k = \frac{k_1}{mc_1^2} c^2 = 1.534 \frac{c^2}{m}\}. \tag{1.180}$$

Note that in the case in question, the Pareto-optimal set depends only on the mass of the body being isolated and does not depend on the intensity β of the shock.

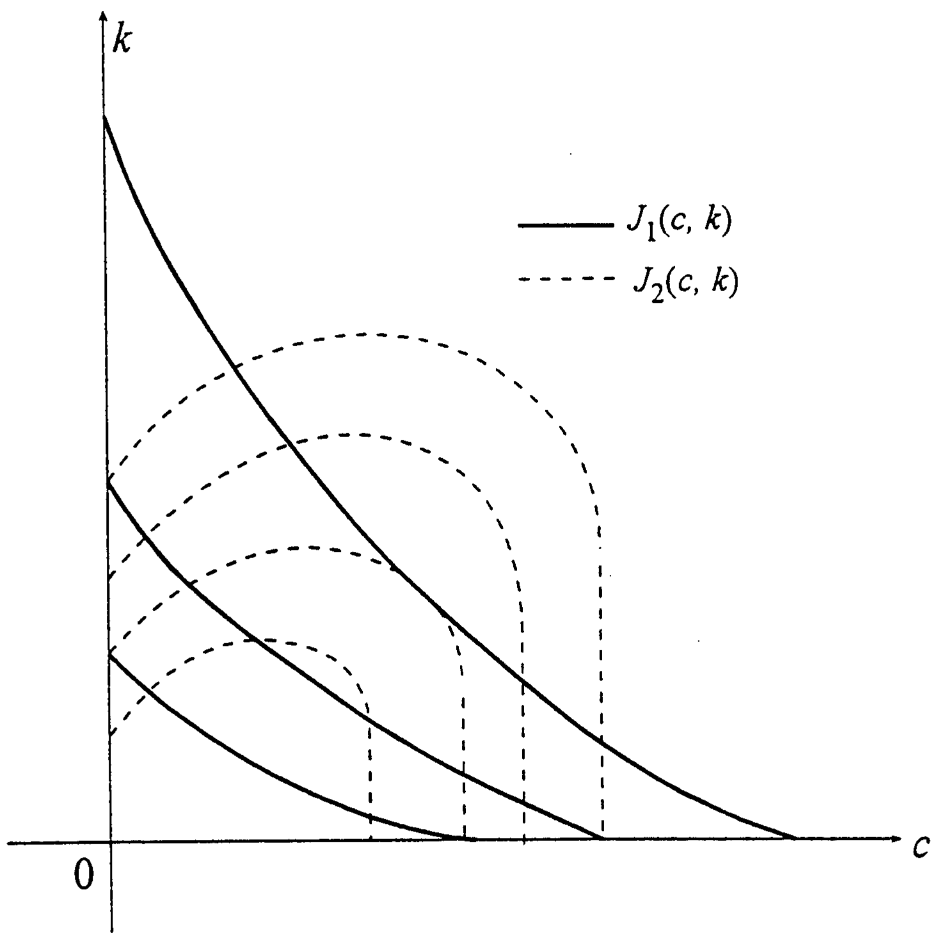


Figure 1-22. Level curves for criteria $J_1(c, k)$ and $J_2(c, k)$.

1.3 METHOD OF PARAMETRIC OPTIMIZATION FOR CALCULATING CHARACTERISTICS OF ISOLATORS.

1.3.1 Specified Disturbance.

Consider the problem of the synthesis of the optimal feedback characteristic for an isolator. Assume the external disturbance is specified (Problem 1.7). Let the system with isolators be governed by the differential equation with the initial conditions of Eq. (1.72). To evaluate the isolation quality the functionals $J_i(\mathbf{u}), i = 1, \ldots, N$, of the form of Eqs. (1.70) or (1.71) will be used. It is required to choose from a specified set U of functions of state variables and, in the case of an active isolation system, time, the optimal characteristic $\mathbf{u}_0 = \mathbf{u}_0(x, t)$ minimizing the functional $J_1(\mathbf{u})$, with the other performance criteria being constrained, i.e.,

$$J_1(\mathbf{u}_0) = \min_{\mathbf{u} \in U}\{J_1(\mathbf{u}) | \quad J_i(\mathbf{u}) \leq D_i, \quad i = 2, \ldots, N\}. \tag{1.181}$$

The exact analytical solution of this problem can be found for only a few systems of rather simple structure. This will be treated in subsequent chapters of this book. The numerical construction of the optimal characteristic $\mathbf{u} = \mathbf{u}_0$ is rather complicated, especially when performance indices are non-additive functionals like that of Eq. (1.70) and/or if the set U of admissible characteristics consists of functions depending only on the state variables, i.e., if only passive isolators are admissible.

Let us discuss an approach that may be useful when solving engineering problems of designing isolation systems. The approach involves replacing the original set U of admissible characteristics by a parametric family $\widetilde{U}$ of functions

$$\widetilde{U} = \{\mathbf{u} : \mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p}), \quad \mathbf{p} \in P\} \tag{1.182}$$

and calculating the desired characteristics by optimization of the functional J_1 with respect to the parameters, provided functionals $J_2, \ldots, J_N$ are constrained. Here, $\mathbf{p}$ is a finite-dimensional vector of parameters, which are the design variables, while P is a fixed admissible set of the parameters.

According to such an approach, the isolator characteristic $\mathbf{u}$ is represented as an explicit function of the state variables, time (for an active isolation system), and the parameter vector $\mathbf{p}$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$ into the right-hand side of the system of Eq. (1.72), we obtain the system of differential equations with parameters and initial conditions

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t, \mathbf{p}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \in [t_0, T], \quad \mathbf{g}(\mathbf{x}, t, \mathbf{p}) = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t, \mathbf{p}), t). \tag{1.183}$$

Performance criteria $J_i(\mathbf{u}), i = 1, \ldots, N$, become functions of the parameters, i.e., the components of the vector $\mathbf{p}$,

$$G_i(\mathbf{p}) = J_i(\tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})). \tag{1.184}$$

The problem of the synthesis of the optimal isolator characteristic is now replaced with the problem of searching for optimal parameters.

1.3.1.1 Problem 1.8. For the system governed by differential equations with parameters and initial conditions of Eq. (1.183), find the parameter vector $\mathbf{p}_0 \in P$ such that

$$G_1(\mathbf{p}_0) = \min_{p \in P}\{G_1(\mathbf{p})| \quad G_i(\mathbf{p}) \le D_i, \quad i = 2, \dots, N\}. \tag{1.185}$$

The set P of admissible values of parameters should be chosen so that $\tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p}) \in U$ for all $\mathbf{p} \in P$, i.e., $\widetilde{U} \subset U$. Refer to Eq. (1.182). Practical calculations will be facilitated if the dimensionality of the vector $\mathbf{p}$ is not very large and the geometry of the domain P is not very complicated.

From an engineering point of view, the specification of the function $\tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$ means that the configuration of the isolation system is prescribed at the outset, whereas its parameters (design variables), e.g., the stiffness and damping coefficients, are to be determined by solving the optimization problem of Eq. (1.185).

Problem 1.8 is the well-known problem of constrained minimization of a multi-variable function. At present, several effective numerical methods of constrained minimization are available, which can be successfully used for solving Problem 1.8. For example, see Wilde (1964), Polak (1971), and Moiseev, et al. (1978). These method are iterative and generate a sequence of admissible values of the parameter vector to be optimized. Under appropriate conditions, this sequence converges to the desired optimal value.

To calculate functions $G_i(\mathbf{p})$, $i = 1, \dots, N$, of Eq. (1.184) it is necessary to integrate the initial-value problem of Eq. (1.183), substitute the result into Eqs. (1.70) and (1.71), and calculate the corresponding values of the functionals J_i, $i = 1, \dots, N$. Thus, the implementation of iterative methods of determining parameters of isolator characteristics on the basis of minimizing the function $G_1(\mathbf{p})$ requires integrating the equations of motion (Eq. 1.183) at each iteration. The choice of a numerical method for searching for the extremum of a multi-variable function and the mode of taking into account the constraints in each particular case depends on the properties of the function $G_1(\mathbf{p})$ to be minimized, the functions $G_i(\mathbf{p})$, $i = 2, \dots, N$, to be constrained, and the geometry of domain P.

For concrete examples of using such an approach for the synthesis of characteristics of isolators, see Viktorov and Larin (1967), Larin (1969), Sevin and Pilkey (1971), and Samsonov (1974). Also, some examples are included in this book.

It is obvious that the characteristic $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p}_0)$, corresponding to the optimal value $\mathbf{p} = \mathbf{p}_0$ of the parameter vector, is not, in general, the solution of Problem 1.7, because the parametric family $\widetilde{U}$ of functions $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$, as a rule, does not cover the whole of the set U of admissible characteristics and does not contain the solution of Problem 1.7.

The proximity of the optimal value of the performance index to be minimized in Problem 1.8 to the minimum of the corresponding functional in Problem 1.7 depends on the adequacy of the

choice of the set $\widetilde{U}$, which replaces the initial set U of admissible characteristics. An *a priori* estimation of the proximity of solutions to Problems 1.7 and 1.8 is rather problematic.

Thus, the approach under discussion is not a strictly substantiated method for an approximate solution of the problem of synthesis of optimal feedback isolator characteristics, but rather a useful practical procedure. It is shown in Chapters 2 and 3 that spring and damper isolators with properly chosen parameters can provide the isolation quality close to the limiting capabilities for objects mounted on a rectilinearly moving base or having a fixed rotation axis. It can then be reasoned that the widespread use of passive isolation elements, e.g., combinations of elastic and dissipative elements with power law characteristics, with properly chosen parameters may allow achieving isolation quality close to the limiting capabilities for more complex systems.

1.3.1.2 <u>Practical Scheme for the Design of Near-Optimal Shock and Vibration Isolators.</u> It is possible to take advantage of the knowledge of the limiting isolation capabilities (Section 1.2.2) in the calculation of optimal feedback isolator characteristics. A multi-stage scheme is described next.

Stage 1. Having specified a function $\mathbf{u} = \tilde{\mathbf{u}}^{(0)}(x, p)$, solve the parametric optimization Problem 1.8. Save in the computer memory the values of components of the optimal parameter vector $\mathbf{p}_0$ and the corresponding value of the optimization criterion $G_1(\mathbf{p}_0)$.

Stage 2. Compare the value $G_1(\mathbf{p}_0)$ with the absolute minimum of the performance index J_1 resulting from the solution of the problem of limiting isolation capabilities.

Stage 3. If the difference obtained is satisfactory to the designer, the characteristic $\mathbf{u} = \tilde{\mathbf{u}}^{(0)}(\mathbf{x}, \mathbf{p}_0)$ is accepted and this completes the calculation. Otherwise, the designer should choose another function, $\mathbf{u} = \tilde{\mathbf{u}}^{(1)}(\mathbf{x}, \mathbf{p})$, corresponding to another structure of the isolation system and go back to Stage 1 replacing $\mathbf{u} = \tilde{\mathbf{u}}^{(0)}(\mathbf{x}, \mathbf{p})$ with $\tilde{\mathbf{u}}^{(1)}(\mathbf{x}, \mathbf{p})$.

An important characteristic of this computer-aided calculation of isolator properties is the designer-computer dialogue.

1.3.2 Unspecified Disturbance.

Consider now an approach using the parametric optimization for the design of isolation systems in the case where the information concerning external disturbances is incomplete. Assume that only a set V of possible disturbances is specified.

1.3.2.1 <u>Problem 1.9</u>. Let the motion of the system with isolators be governed by the vector differential equation of Eq. (1.54) with the initial conditions of Eq. (1.55), while the isolation quality is evaluated by functionals $J_i(\mathbf{u}, \mathbf{v}), i = 1, \ldots, N$, of the form of Eqs. (1.56) or (1.57). It is required to find from the specified set U of functions of the state variables and, for the active isolation system, time, the optimal characteristic $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}, t)$ that minimizes the guaranteed (not exceedable under the least favorable of the possible external disturbances) value of the functional $J_1(\mathbf{u}, \mathbf{v})$, provided the guaranteed values of the other performance criteria are constrained, i.e.

$$\widetilde{J}_1(\mathbf{u}_0) = \min_{\mathbf{u}\in U}\{\widetilde{J}_1(\mathbf{u})| \quad \widetilde{J}_i(\mathbf{u}) \le D_i, \quad i = 2, \ldots, N\}, \tag{1.186}$$

$$\widetilde{J}_i(\mathbf{u}) = \max_{\mathbf{v}\in V} J_i(\mathbf{u},\mathbf{v}), \quad i = 1,\ldots,N.$$

If one changes the initial set U for the parametric family of Eq. (1.182) of functions $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x},t,\mathbf{p})$, Problem 1.9 is replaced with the following problem of optimizing the isolator's parameters.

1.3.2.2 Problem 1.10. Let the system with isolators be governed by vector differential equations with parameters and initial conditions as

$$\begin{gathered}\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x},\mathbf{v},t,\mathbf{p}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \in [t_0,T], \\ \mathbf{h}(\mathbf{x},\mathbf{v},t,\mathbf{p}) = \mathbf{f}(\mathbf{x},\tilde{\mathbf{u}}(\mathbf{x},t,\mathbf{p}),\mathbf{v},t) \quad \mathbf{v}\in V, \quad \mathbf{p}\in P.\end{gathered} \tag{1.187}$$

Find the parameter vector $\mathbf{p}_0 \in P$ such that

$$\begin{gathered}\widetilde{G}_i(\mathbf{p}_0) = \min\nolimits_{\mathbf{p}\in P}\{\widetilde{G}_1(\mathbf{p})|\widetilde{G}_i(\mathbf{p}) \le D_i, \quad i = 2,\ldots,N\}, \\ \widetilde{G}_i(\mathbf{p}) = \max\nolimits_{\mathbf{v}\in V} G_i(\mathbf{p},\mathbf{v}), \quad G_i(\mathbf{p},\mathbf{v}) = J_i(\tilde{\mathbf{u}}(\mathbf{x},t,\mathbf{p}),\mathbf{v}), \quad i = 1,\ldots,N.\end{gathered} \tag{1.188}$$

Like Problem 1.8, Problem 1.10 is reduced to the constrained minimization of a function of a finite number of variables $(G_1(\mathbf{p}))$. This minimization can be carried out by available numerical methods of nonlinear programming. However, unlike Problem 1.8, Problem 1.10 requires solving N variational problems for the maximization, with respect to all possible external disturbances $\mathbf{v}(t) \in V$, of functionals $G_i(\mathbf{p},\mathbf{v}) = J_i(\tilde{\mathbf{u}}(\mathbf{x},t,\mathbf{p}),\mathbf{v})$, $i = 1,\ldots,N$, to calculate functions $\widetilde{G}_i(\mathbf{p}), i = 1,\ldots,N$, at each value of the parameter vector $\mathbf{p}$. Since the initial conditions for Eq. (1.187) are assumed to be given, while external disturbances are, as a rule, modeled by functions depending on time only, these variational problems are problems of searching for optimal open-loop controls, with no constraints being imposed on the state variables. Such problems can be solved by methods discussed in Section 1.2.2 devoted to the problem of limiting isolation capabilities.

The solution of Problem 1.10 is simplified if the set V of possible external disturbances is represented by a parametric family of functions as follows:

$$V = \{\mathbf{v} : \mathbf{v} = \tilde{\mathbf{v}}(t,\mathbf{q}), \quad \mathbf{q}\in Q\}. \tag{1.189}$$

where $\mathbf{q}$ is a finite-dimensional parameter vector, and Q is a fixed domain. In this case, to calculate functions $\widetilde{G}_i(\mathbf{p})$, $i = 1,\ldots,N$, defined in Eq. (1.188), it is necessary to calculate the maxima (over $\mathbf{q}\in Q$) of functions of a finite number of variables rather than to solve an optimal control problem. However, if the functions $\mathbf{v} = \tilde{\mathbf{v}}(t,\mathbf{q}), \mathbf{q}\in Q$, o not cover the initial set V of external disturbances, the replacement of the set V by the parametric family of Eq. (1.189) is hardly advisable. If $\tilde{\mathbf{u}}(\mathbf{x},t,\mathbf{p}) \in U$ for all $\mathbf{p}\in P$, then the solution of Problem 1.10 leads to the calculation of the guaranteed (not exceedable even under the least favorable external disturbance $\mathbf{v}(t)\in V$) value of the initial optimization criterion $J_1(\mathbf{u},\mathbf{v})$ (see statement of Problem 1.9), provided guaranteed values of functionals $J_2(\mathbf{u},\mathbf{v}),\ldots,J_N(\mathbf{u},\mathbf{v})$ do not exceed specified admissible limits.

If the designer knows the operation characteristics of the object being isolated and the solution of Problem 1.9, it is possible to evaluate the applicability of the isolator with the characteristic $\mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p}_0)$ for the protection of the object against external disturbances of the set V. However, the change of the initial set of possible external disturbances for the parametric family of functions of Eq. (1.189) can lead to underestimated values for functions $G_i(\mathbf{p})$ in Eq. (1.188) and, hence, erroneous conclusions concerning the applicability of the isolator with the characteristic $\mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p}_0)$. This is explained by the fact that functions $\mathbf{v} = \mathbf{v}(t, \mathbf{q})$, as a rule, do not cover all of the set V as $\mathbf{q}$ runs through the domain Q and do not contain the worst disturbance for each isolator characteristic given by

$$\mathbf{u}(x, t) = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p}), \quad \mathbf{p} \in P. \tag{1.190}$$

The methods of parametric optimization of characteristics of isolators described above require integrating differential governing equations of motion at each step of the iteration process of searching for the extremum. This makes the optimization procedure rather time consuming, especially if the initial approximation of the parameters is chosen far from the optimum.

1.3.3 Indirect Optimization.

As a final topic in this section, we introduce a method of parametric optimization that does not require integrating equations of motion at each iteration. However, when using this method, one has to know the solution to the problem of limiting isolation capabilities. This approach was proposed in Sevin and Pilkey (1971).

Let the optimal open-loop characteristic $\mathbf{u} = \mathbf{u}_0(t)$ of the isolation system and the corresponding state trajectory $\mathbf{x} = \mathbf{x}_{u_0}(t)$ of the system with isolators be known as a result of solving the problem of limiting isolation capabilities. To find parameters of the characteristic $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$ according to the method in question, minimize the function of a finite number of variables given by

$$\Delta(\mathbf{p}) = \|\tilde{\mathbf{u}}(\mathbf{x}_{u_0}(t), t, \mathbf{p}) - \mathbf{u}_0(t)\|, \quad \mathbf{p} \in P. \tag{1.191}$$

Here, $\|\cdot\|$ means a norm in the space of functions of time. The particular choice of the norm depends on specific features of the problem. If the parametric family of Eq. (1.182) of characteristics $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$ contains the characteristic (corresponding to a certain value $\mathbf{p}_0$ of the parameter vector $\mathbf{p}$) ensuring the motion of the system along the trajectory $\mathbf{x} = \mathbf{x}_{u_0}(t)$, then $\Delta(\mathbf{p}_0) = 0$ and the components of the vector $\mathbf{p}_0$ are the desired optimal parameters. However, as a rule, the parametric family of Eq. (1.182) does not contain the parameters that can provide the limiting performance of the isolation system and the minimum value of the function of Eq. (1.191) is greater than zero.

Calculations carried out in Sevin and Pilkey (1971) and Afimiwala and Mayne (1974) show that if the minimum value of the function $\Delta(\mathbf{p})$ is comparatively small, then the parameters determined with this approach provide isolation quality close to the limiting isolation capabilities. However, if the minimum value of $\Delta(\mathbf{p})$ is not small, it is difficult to determine whether the parameters minimizing the function of Eq. (1.191) are close to the optimal parameters of the characteristic

$\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$ resulting from the solution of Problems 1.8 or 1.10. In addition, the method under consideration does not take into account the constraints imposed on the performance criteria other than the optimization criterion.

The application of this method seems to be limited. It can be effective in preliminary calculations aimed, for instance, at finding out whether the chosen structure of the isolation system corresponding to the parametric family of Eq. (1.182) of isolator characteristics is able to provide behavior of the system close to the limiting performance. The parameters obtained by the minimization of the function of Eq. (1.191) can be used as initial approximations to the optimal parameters of the characteristic $\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}, t, \mathbf{p})$. However, for the fine adjustment of the parameters, it is recommended that the method associated with the solution of Problems 1.8 or 1.10 be used.

SECTION 2

OPTIMAL PROTECTION OF RECTILINEARLY MOVING SYSTEMS FROM AN INSTANTANIOUS IMPACT

2.1 PROBLEM FORMULATION.

Rectilinear motion is the simplest type of mechanical motion. Restriction to this type of motion enables one to investigate the dynamics of a system in more detail and to obtain analytical (rather than numerical) solutions for a number of optimal isolation problems. Moreover, rectilinear motion is a widespread occurrence in engineering systems, so that the computational formulas and algorithms presented in this section are appropriate for practical calculations of isolation systems.

Consider a body attached to a base by means of a shock or vibration isolator (Fig. 2.1). In what follows, the body or object being isolated will often be referred as simply "the body" or "the object". The base moves rectilinearly. The body being isolated can move relative to the base so that the line of motion on the body coincides with that of the base. Both the base and the object being isolated are regarded as absolutely rigid bodies. The control force f (the isolator characteristic) is assumed to depend on the displacement x of the body being isolated relative to the base, its relative velocity $\dot{x}$, and (in the case of an active isolator) time t. The actual form of the function $f(x, \dot{x}, t)$ is determined by the structure of the isolating device. Denote by m and M the masses of the body being isolated and the base, respectively, and by y, the displacement of the base relative to an inertial reference frame. If the force $\sigma(t)$ applied to the base by the environment is specified, then the motion of the system is governed by the set of differential equations

$$M\ddot{z} + m(\ddot{x} + \ddot{z}) = \sigma(t), \quad m(\ddot{x} + \ddot{z}) = f(x, \dot{x}, t). \tag{2.1}$$

These equations describe the motion of the base and the body being isolated in the case of a dynamic external disturbance. (Remember that the external disturbance is said to be dynamic if it is specified as a force applied to a member of the system with isolators. See Section 1.1.4. In the case under consideration, the external force is applied to the base.) Eliminate the variable z from Eq. (2.1) to find

$$\ddot{x} - \frac{f(x, \dot{x}, t)}{\mu} = -\frac{\sigma(t)}{M}, \quad \mu = \frac{Mm}{M + m}. \tag{2.2}$$

In the case of a prescribed base displacement (kinematic external disturbance), where the time history of the acceleration $\ddot{z}(t)$ of the base is known, the equation governing the relative motion of the body being isolated is given by

$$\ddot{x} - \frac{f(x, \dot{x}, t)}{m} = -\ddot{z}(t). \tag{2.3}$$

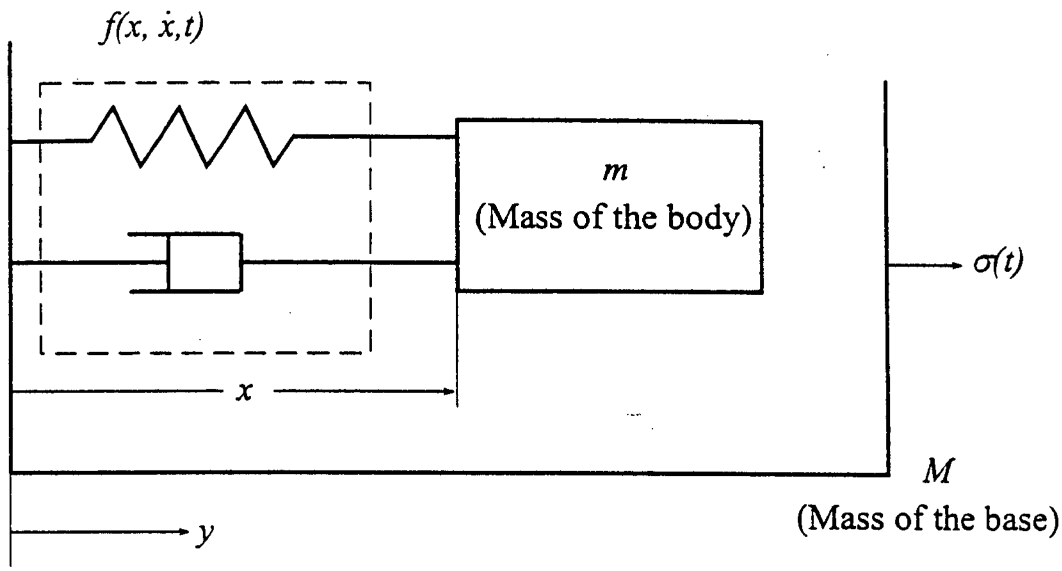


Figure 2-1. The base and the body to be isolated moving rectilineary.

This corresponds to a single degree of freedom (SDOF) system. It follows from Eqs. (2.2) and (2.3) that for both types of external disturbances the governing equation can be expressed as

$$\ddot{x} + u(x, \dot{x}, t) = F(t). \tag{2.4}$$

Here, $u(x, \dot{x}, t) = -f(x, \dot{x}, t)/m$ and $F(t) = -\ddot{z}(t)$ for the kinematic external disturbance, while for the dynamic disturbance $u(x, \dot{x}, t) = -f(x, \dot{x}, t)/\mu$ and $F(t) = -\sigma(t)/M$.

The function $u(x, \dot{x}, t)$, apart from a constant factor, coincides with the isolator characteristic $f(x, \dot{x}, t)$, while, similarly, the function $F(t)$ is equal to the quantity describing the external disturbance (the force $\sigma(t)$ or the acceleration $\ddot{z}(t)$). In what follows, we will refer to either $u(x, \dot{x}, t)$ or $f(x, \dot{x}, t)$ as the isolator characteristic, and either $F(t)$ or $\sigma(t)$ as the external disturbance.

To determine uniquely the motion of the system from Eq. (2.4) prescribe initial conditions at some time instant $t = t_0$.

$$x(t_0) = x^0, \quad \dot{x}(t_0) = \dot{x}^0. \tag{2.5}$$

This chapter is devoted to the optimal protection of objects against shock-type disturbances. The most important criteria characterizing the quality of shock isolation (see Section 1.1.3) are the maximum absolute value of the relative displacement of the body being isolated

$$J_1(u, F; x^0, \dot{x}^0) = \max_{t \in [t_0, \infty)} |x(t)| \tag{2.6}$$

and the maximum absolute value of the acceleration of the body with respect to an inertial reference frame (absolute acceleration)

$$J_2(u, F; x^0, \dot{x}^0) = \max_{t \in [t_0, \infty)} |u(x(t), \dot{x}(t), t)|. \tag{2.7}$$

In these expressions $x(t)$ is the solution of the differential equation of Eq. (2.4) subject to the initial conditions of Eq. (2.5).

The criteria J_1 and J_2 are functionals of the external disturbance $F(t)$ and the isolator characteristic $u(x, \dot{x}, t)$, and, in addition, are functions of the initial coordinate x^0 and velocity $\dot{x}^0$ of the body being isolated.

Consider typical optimum isolation problems for rectilinearly moving systems under a known external disturbance $F(t)$. For the sake of generality, assume that the initial state variables of Eq. (2.5) are not completely determined but belong to a specified domain G_1 in the state space, i.e.,

$$(x^0, \dot{x}^0) \in G_1. \tag{2.8}$$

Equation (2.8) corresponds to the exact specification of the initial state if the domain G_1 contains only a single point.

2.1.1.1 Problem 2.1. Let the relative motion of the object being isolated be governed by Eq. (2.4). It is required to find, from among a specified class Y of functions $u(x, \dot{x}, t)$, the optimal isolator characteristic $u_0(x, \dot{x}, t)$ such that

$$\max_{(x^0,\dot{x}^0)\in G_1} J_1(u_0, F; x^0, \dot{x}^0) = \min_{u\in Y} \max_{(x^0,\dot{x}^0)\in G_1} J_1(u, F; x^0, \dot{x}^0), \tag{2.9}$$

$$\max_{(x^0,\dot{x}^0)\in G_1} J_2(u_0, F; x^0, \dot{x}^0) \leq U.$$

This statement corresponds to the requirement of minimizing the relative displacement of the object being isolated, provided the force transmitted to it does not exceed the admissible level U. This constraint is intended to ensure the reliable functioning of the object.

2.1.1.2 Problem 2.2. The relative motion of the object being isolated is governed by Eq. (2.4). It is required to find, from among a specified class Y of functions $u(x, \dot{x}, t)$, the optimal isolator characteristic $u^0(x, \dot{x}, t)$ such that

$$\max_{(x^0,\dot{x}^0)\in G_1} J_2(u^0, F; x^0, \dot{x}^0) = \min_{u\in Y} \max_{(x^0,\dot{x}^0)\in G_1} J_2(u, F; x^0, \dot{x}^0), \tag{2.10}$$

$$\max_{(x^0,\dot{x}^0)\in G_1} J_1(u^0, F; x^0, \dot{x}^0) \leq D.$$

This problem reflects the requirement of minimizing the maximum force transmitted to the object under a constraint on the allowable displacement relative to the base.

The choice of the class Y of allowable isolator characteristics is based on practical implementability of these characteristics or is determined by the needs of the theoretical analysis (as in the problem of the limiting isolation capabilities considered below) .

For solving Problems 2.1 and 2.2, only certain information concerning the motion of the body relative to the base and the time history of the external disturbance is necessary. The knowledge of the position and velocity of the base with respect to an inertial reference frame is not needed. The relative motion of the body is described by Eq. (2.4) for either kinematic or dynamic external disturbances. Therefore, problems of optimal protection of rectilinearly moving objects from disturbances of either type are mathematically equivalent.

In this section, we assume that the external disturbance is an impulse (instantaneous shock) occurring at the initial time instant $t = 0$. Then

$$F(t) = \beta\delta(t), \tag{2.11}$$

where δ is the Dirac delta-function and β is the magnitude of the impulse (the shock intensity). The quantity β is equal numerically to the magnitude of the jump in the velocity of the object being isolated resulting from the impulse. We also assume (except in Section 2.3.7) that at the instant of shock, the body being isolated is in the state of rest relative to the base, so that

$$x(0) = 0, \quad \dot{x}(0) = 0. \tag{2.12}$$

In this case, the domain G_1 in Eq. (2.8) consists of a single point: $G_1 = \{0, 0\}$.

For compactness, in what follows, we omit some of the arguments of performance criteria of Eqs. (2.6) and (2.7).

Equation (2.4) with the initial conditions of Eq. (2.12) and the external disturbance of Eq. (2.11) is equivalent to the initial-value problem

$$\ddot{x} + u(x, \dot{x}, t) = 0, \quad x(0) = 0, \quad \dot{x}(0) = \beta. \tag{2.13}$$

This equivalence reflects the fact that as a result of the shock disturbance the body acquires a finite relative velocity equal to β.

In solving the problems, we will show that at the optimal isolator characteristic, the criterion subject to the constraint assumes the maximum allowable value (D or U), while the value of the criterion to be minimized monotonically decreases as the parameter D or U increases. Hence, according to Theorem 1.1, Problems 2.1 and 2.2 are reciprocals of each other, and having solved one of the problems, one can easily obtain the solution to the other. At the outset, we will consider Problem 2.2.

2.2 LIMITING ISOLATION CAPABILITIES .

Let us solve the problem of limiting isolation capabilities for the system described by the differential equation with initial conditions of Eq. (2.13). From the class Y of piecewise continuous, continuous on the right at discontinuity points, functions of time $u(t)$, we wish to find the optimal isolator characteristic $u^0(t)$ such that

$$J_2(u^0) = \max_{t\in[0,\infty)} |u^0(t)| = \min_{u\in Y} J_2(u) = \min_{u\in Y} \max_{t\in[0,\infty)} |u(t)| \tag{2.14}$$

$$J_1(u^0) = \max_{t\in[0,\infty)} |x(t)| \le D. \tag{2.15}$$

The solution to this problem provides the minimum of the maximum absolute value of the force transmitted to the body being isolated. This characterizes the limiting performance of shock isolation for a specified intensity of loading and a constraint on the relative displacement.

Without loss of generality, set $\beta = 1$ and $D = 1$ in Eqs. (2.13) and (2.15). This corresponds to the use of the dimensionless (primed) variables

$$x' = \frac{x}{D}\mathrm{sign}(\beta), \quad t' = \frac{|\beta|}{D}t, \quad u' = \frac{D\beta}{|\beta|^3}u. \tag{2.16}$$

In what follows, the dimensionless variables will be used, unless indicated otherwise. However, the primes will be omitted.

It is readily shown that

$$\max_{t\in[0,\infty)} \left|u^0(t)\right| \geq 0.5. \tag{2.17}$$

According to Cauchy's formula, the initial-value problem of Eq. (2.13) with $\beta = 1$ is equivalent to the integral equation

$$x(t) = t - \int_0^t (t-\tau)u(x(\tau), \dot{x}(\tau), \tau)d\tau. \tag{2.18}$$

This can be verified by differentiation. If u is a function of t alone, i.e., $u = u(t)$, then

$$x(t) = t - \int_0^t (t-\tau)u(\tau)d\tau \tag{2.19}$$

is the solution to the initial-value problem of Eq. (2.13).

Assume that the inequality of Eq. (2.17) is not valid, i.e., the inequality $|u^0(t)| < 0.5$ holds for any $t \in [0, \infty)$. This implies that $u^0(t) < 0.5$ for any $t \in [0,\infty)$ and, hence,

$$-u^0(t) > -0.5, \qquad t \in [0, \infty). \tag{2.20}$$

The substitution of $u^0(\tau)$ for $u(\tau)$ into Eq. (2.19), with allowance for Eq. (2.20), leads to the inequality

$$x(t) > t - 0.5\int_0^t (t-\tau)d\tau. \tag{2.21}$$

or

$$x(t) > t - \frac{t^2}{4}. \tag{2.22}$$

It follows from Eq. (2.22) that

$$\max_{t\in[0,\infty)} x(t) > \max_{t\in[0,\infty)} \left(t - \frac{t^2}{4}\right). \tag{2.23}$$

On calculating the maximum on the right-hand side of Eq. (2.23) we obtain

$$\max_{t\in[0,\infty)} x(t) > 1. \tag{2.24}$$

Thus, if $| u^0(t) | < 0.5$ for $t \in [0, \infty)$, then the inequality in Eq. (2.15) is not satisfied in the case of $D = 1$. The contradiction proves the inequality of Eq. (2.17).

It is readily verified that any control given by

$$u^0(t) = \begin{cases} 0.5, & \text{if } 0 \le t < 2 \\ \bar{u}(t) : |\bar{u}(t)| \le 0.5 \quad \text{and} \quad |x(t)| \le 1, & \text{if } t \ge 2 \end{cases} \tag{2.25}$$

leads to the relationships

$$\max_{t\in[0,\infty)} |u^0(t)| = 0.5, \quad \max_{t\in[0,\infty)} |x(t)| = x(2) = 1. \tag{2.26}$$

The expression of Eq. (2.25) means that in the time interval $0 \le t < 2$, that is, until the velocity $\dot{x}$ vanishes for the first time, the control $u^0(t)$ is constant and is equal to 0.5, while for $t \ge 2$ it can be any piecewise continuous function satisfying the inequalities indicated in Eq. (2.25). It follows from Eqs. (2.17) and (2.26) that any function $u^0(t)$ of the form of Eq. (2.25) is the optimal control solving the problem of optimal isolation capabilities. The optimal control is unique in the interval $0 \le t < 2$ and nonunique for $t \ge 2$. Such nonuniqueness is a typical and essential feature of optimal control problems with nonadditive maximum-type functionals, a category to which many problems of optimal isolation belong.

For the control $u^0(t)$ of the form of Eq. (2.25), the time history of the relative displacement and velocity of the body being isolated in the interval $0 \le t \le 2$ is given by

$$x(t) = t - t^2/4, \quad \dot{x}(t) = 1 - t/2. \tag{2.27}$$

The relationships of Eq. (2.26) determine the limiting capabilities of protection of the body being isolated against shock disturbances. It is impossible in principle to make the maximum absolute value of the force transmitted to the body less than 0.5 if the shock imparts the velocity $\dot{x}(0) = 1$ to the body and the maximum displacement does not exceed unity.

Turn back to the original dimensional variables in Eqs. (2.25) and (2.26) according to Eq. (2.16) to obtain

$$u^0(t) = \begin{cases} 0.5\frac{\beta^2}{D}, & \text{if } 0 \le t < 2D/|\beta| \\ \bar{u}(t) : |\bar{u}(t)| \le 0.5\frac{\beta^2}{D} \quad \text{and} \quad |x(t)| \le D, & \text{if } t \ge 2D/|\beta| \end{cases} \tag{2.28}$$

$$J_2(u^0) = \max_{t\in[0,\infty)} |u^0(t)| = 0.5\frac{\beta^2}{D}, \quad J_1(u^0) = \max_{t\in[0,\infty)} |x(t)| = D.$$

The relations of Eq. (2.28) express the solution of the problem of Eqs. (2.14) and (2.15) for arbitrary values of the parameters β and D.

It is seen from Eq. (2.28) that for the optimal isolator characteristic $u = u^0$, the maximum absolute value of the displacement of the body being isolated assumes the maximum allowable value D and the corresponding optimal value of the performance index to be minimized, $J_2(u^0)$, monotonically decreases as the parameter D increases. According to Theorem 1.1 of Section 1.2.1, the problem of Eqs. (2.14) and (2.15) of the limiting capabilities of minimizing the maximum

absolute value of the load transmitted to the body, provided its displacement is constrained, is the reciprocal (the dual) of the problem of the limiting capabilities of minimizing the maximum absolute value of the displacement under the constraint on the transmitted load. The application of Theorem 1.1 leads to the solution of the latter problem in the form

$$u_0(t) = \begin{cases} U, & \text{if} \quad 0 \le t < \frac{|\beta|}{U} \\ \tilde{u}(t) : |\tilde{u}(t)| \le U \quad \text{and} \quad |x(t)| \le 0.5\frac{\beta^2}{U}, & \text{if} \quad t \ge \frac{|\beta|}{U} \end{cases} . \tag{2.29}$$

$$J_1(u_0) = 0.5\frac{\beta^2}{U}, \quad J_2(u_0) = U.$$

2.3 OPTIMIZATION OF PARAMETERS OF PASSIVE ISOLATORS WITH STIFFNESS AND DAMPING ELEMENTS.

2.3.1 Statement of the Problem.

When designing systems for protecting objects against shock and vibration, it is often desirable to use only passive isolators whose characteristics do not depend on time explicitly (see Section 1.1.4).

Let us consider isolators with the characteristic $u(x, \dot{x})$ given by

$$u(x, \dot{x}) = \psi(c, \dot{x})\text{sign}(\dot{x}) + \varphi(k, x)\text{sign}(x). \tag{2.30}$$

Here, $\psi(c, \dot{x})$ and $\varphi(k, x)$ are non-negative functions describing the dependence of the damping force and the stiffness element restoring force on the relative velocity and displacement of the body being isolated. The parameters $c \ge 0$ and $k \ge 0$ are constant damping and stiffness factors, respectively. The function $\text{sign}(\zeta)$ is defined as

$$\text{sign}(\zeta) = \begin{cases} -1, & \text{if } \zeta < 0 \\ 0 & \text{if } \zeta = 0 \\ 1 & \text{if } \zeta > 0 \end{cases} . \tag{2.31}$$

The class of functions of Eq. (2.30) covers a rather wide variety of isolator characteristics commonly used in practice. For example, for the familiar isolator with a linear spring and a linear damper, we have $u(x, \dot{x}) = c\dot{x} + kx$, where c and k are the damping and stiffness coefficients, respectively. The characteristic of this isolator can be expressed in the form of Eq. (2.30) as $u(x, \dot{x}) = c|\dot{x}|\text{sign}(\dot{x}) + k|x|\text{sign}(x)$, and, hence, $\psi(c, \dot{x}) = c\,|\dot{x}|$ and $\varphi(k, x) = k\,|x|$. The characteristic $u(x, \dot{x}) = c|\dot{x}|\dot{x} + kx$ of the isolator with a linear spring and a quadratic-law damper can be represented in the form of Eq. (2.30) with $\psi(c, \dot{x}) = c\dot{x}^2$ and $\varphi(k, x) = k\,|x|$. Consider now particular cases of a damper with $\psi(c, \dot{x}) = c$ and a stiffness element with $\varphi(k, x) = k$. The damper with $\psi(c, \dot{x}) = c$ is a Coulomb damper (dry-friction damper). It resists motion with a force $-c\,\text{sign}(\dot{x})$ which is constant in magnitude and is directed against the velocity of the body.

For more detail, see Section 2.3.7. The stiffness element with $\varphi(k,x) = k$ resists displacement with a force $-k\text{sign}(x)$ that is constant in magnitude and is directed against the displacement. The displacement is measured relative to the equilibrium position at which the resistance force is zero. We will refer to this element as *bang-bang spring.* This term is formed by analogy with the term bang-bang control in control theory. The bang-bang control switches between constant values. In a similar manner, the resistance force generated by a bang-bang spring switches between two values, differing in sign, when the body passes through the equilibrium position.

The control force corresponding to the characteristic of Eq. (2.30) contains only conservative and dissipative components. In practice, such a characteristic is formed of simple stiffness (elastic) and damping elements, without external power supplies and control units. This makes the system less expensive and more reliable in operation.

For the isolator characteristic of Eq. (2.30), the initial value problem of Eq. (2.13) becomes

$$\ddot{x} + \psi(c,\dot{x})\text{sign}(\dot{x}) + \varphi(k,x)\text{sign}(x) = 0, \tag{2.32}$$

$$x(0) = 0, \quad \dot{x}(0) = \beta.$$

It is convenient to introduce new notation for the performance criteria of Eqs. (2.6) and (2.7)

$$J_1(u) = I_1(c,k) = \max_{t\in[0,\infty)} |x(t)|\,, \tag{2.33}$$

$$J_2(u) = I_2(c,k) = \max_{t\in[0,\infty)} |u(x(t),\dot{x}(t))| = \max_{t\in[0,\infty)} |\ddot{x}(t)| =$$

$$= \max_{t\in[0,\infty)} |\psi(c,\dot{x}(t))\text{sign}(\dot{x}(t)) + \varphi(k,x(t))\text{sign}(x(t))| \tag{2.34}$$

reflecting the dependence of the criteria on the parameters c and k.

Let us reformulate Problems 2..1 and 2.2 for the case where the motion of the object being isolated is governed by the initial-value problem of Eq. (2.32) with β being specified.

2.3.1.1 Problem 2.3. Find the parameters (design variables) $c_0 \geq 0$ and $k_0 \geq 0$ such that

$$\begin{array}{c} I_1(c_0,k_0) = \min_{c\geq 0,k\geq 0} I_1(c,k) \\ I_2(c_0,k_0) \leq U \end{array} . \tag{2.35}$$

2.3.1.2 Problem 2.4. Find the parameters $c^0 \geq 0$ and $k^0 \geq 0$ such that

$$I_2(c^0, k^0) = \min_{c \geq 0, k \geq 0} I_2(c, k) \quad I_1(c^0, k^0) \leq D \tag{2.36}$$

Here, the class Y appearing in Problems 2.1 and 2.2 is the two-parameter family of functions of Eq. (2.30), c and k being the parameters; the set G_1 of possible initial values of the phase variables consists of a single point $(x^0, \dot{x}^0) = (0, \beta)$.

In the next section we establish, under certain assumptions for the functions $\psi(c, \dot{x})$ and $\varphi(k, x)$, a number of the performance criteria properties that can help facilitate solution of optimization Problems 2.3 and 2.4.

2.3.2 Performance Criteria Properties.

2.3.2.1 Basic Assumptions. Let us make the following assumptions concerning functions $\psi(c, \dot{x})$ and $\varphi(k, x)$.

Assumption 1. Functions $\psi(c, \dot{x})$ and $\varphi(k, x)$ are nonnegative, continuous, and continuously differentiable everywhere over $\dot{x} \in (-\infty, \infty)$ or $x \in (-\infty, \infty)$, except, perhaps, $\dot{x} = 0$ or $x = 0$.

Assumption 2. Functions $\psi(c, \dot{x})$ and $\varphi(k, x)$ are even in the variables $\dot{x}$ and x, respectively, i.e.,

$$\psi(c, \dot{x}) = \psi(c, -\dot{x}), \quad \varphi(k, x) = \varphi(k, -x); \tag{2.37}$$

Assumption 3.

$$\psi(c, \dot{x}_1) \geq \psi(c, \dot{x}_2), \quad \text{if} \quad |\dot{x}_1| > |\dot{x}_2| \, ; \tag{2.38}$$

$$\varphi(k, x_1) \geq \varphi(k, x_2), \quad \text{if} \quad |x_1| > |x_2| \, ; \tag{2.39}$$

$$\begin{array}{llll} \psi(c, \dot{x}) \neq 0, & \text{if} \quad c \neq 0 & \text{and} & \dot{x} \neq 0; \\ \varphi(k, x) \neq 0, & \text{if} \quad k \neq 0 & \text{and} & x \neq 0; \end{array} \tag{2.40}$$

Assumption 4.

$$\begin{array}{lll} \frac{\partial \psi(c, \dot{x})}{\partial c} > 0, & \text{if} & \dot{x} \neq 0; \\ \frac{\partial \varphi(k, x)}{\partial k} > 0, & \text{if} & x \neq 0; \end{array} \tag{2.41}$$

$$\psi(0, \dot{x}) = 0, \quad \varphi(0, x) = 0; \tag{2.42}$$

$$\begin{array}{ll} \lim_{c \to \infty} \psi(c, \dot{x}) = \infty, & \text{if} \quad \dot{x} \neq 0; \\ \lim_{k \to \infty} \varphi(k, x) = \infty, & \text{if} \quad x \neq 0. \end{array} \tag{2.43}$$

Assumption 5.

$$\varphi_k'(k,x)\varphi(k,x) - \varphi_x'(k,x)\int_0^x \varphi_k'(k,\xi)d\xi > 0, \quad \text{if} \quad k \neq 0 \text{ and } x \neq 0. \tag{2.44}$$

In Eq. (2.44), the notation

$$\varphi_k'(k,x) = \frac{\partial\varphi(k,x)}{\partial k}, \qquad \varphi_x'(k,x) = \frac{\partial\varphi(k,x)}{\partial x}, \qquad \varphi_k'(k,\xi) = \frac{\partial\varphi(k,\xi)}{\partial k}. \tag{2.45}$$

is used. Similar notation for partial derivatives will be used in what follows in this section.

Assumption 1 implies that the restoring and damping forces are directed opposite to the displacement x or velocity $\dot{x}$. According to Assumption 2, the stiffness and damping characteristics are symmetrical, and the intensities (absolute values) of the restoring and damping forces depend on the displacement and velocity absolute values and are independent of their directions. In Assumption 3, inequalities of Eqs. (2.38) and (2.39) mean that the intensity of restoring and damping effects increases with the growth of $|x|$ and $|\dot{x}|$. The conditions of Eq. (2.40) mean that dead zones are not allowed for the stiffness and damping elements. It follows from the inequalities of Eq. (2.41) of Assumption 4 that absolute values of the damping and restoring forces increase with the growth of the damping and stiffness factors. The relationships of Eq. (2.42) mean that zero values of c or k correspond to the absence of a damper or a stiffness element. The limiting relations of Eq. (2.43) reflect the possibility of getting arbitrarily large restoring and damping forces by choosing sufficiently large stiffness and damping factors.

As might be expected, Assumptions 1 to 4 are satisfied by virtually all isolators used in practice. Assumption 5 does not lend itself to a simple physical interpretation and seems to be rather artificial. However, it implies some useful properties that can significantly facilitate the search for the isolator optimal parameters. Assumption 5 is satisfied by a wide class of stiffness elements used in isolators, e.g., by all springs with power law characteristics: $\varphi(k,x) = k\,|x|^n$ where n is a nonnegative number. For such springs, the left-hand side of Eq. (2.44) is equal to $k\,|x|^{2n}/(n+1) \geq 0$. The role of Assumption 5 will be clear from some of the following proofs.

2.3.2.2 Propositions and Proofs. Let us formulate and prove a number of propositions related to performance criteria $I_1(c,k)$ and $I_2(c,k)$ given by Eqs. (2.33) and (2.34). In what follows, we sometimes will use $\psi(\dot{x})$ and $\varphi(x)$ instead of $\psi(c,\dot{x})$ and $\varphi(k,x)$, omitting constant arguments c and k for brevity. Remember that we are considering a system for which the motion is governed by Eq. (2.32), that is, the case where the base to which the body being isolated is attached is subject to the instantaneous impact of Eq. (2.11). Hence, the propositions below are valid only for systems of this sort.

2.3.2.3 Proposition 2.1. The maximum absolute value of the displacement of the body being isolated in Eq. (2.33) occurs at the instant t_* of the first local extremum of the function $x(t)$.

2.3.2.4 Proof. Since $x(0) = 0$, the maximum absolute value of the displacement occurs at an instant of local extremum of the function $x(t)$. At this instant, $\dot{x} = 0$. Introduce the function

$$W(x,\dot{x}) = \frac{\dot{x}^2}{2} + \Pi(x), \tag{2.46}$$

where

$$\Pi(x) = \int_0^x \varphi(\xi)\text{sign}(\xi)d\xi. \tag{2.47}$$

From a physics standpoint, W can be interpreted as the total mechanical energy of the system governed by Eq. (2.32), while Π is the potential energy of the stiffness element. Because of Assumption 2, the potential energy can be represented as

$$\Pi(x) = \Phi(|x|) = \int_0^{|x|} \varphi(\xi)d\xi, \tag{2.48}$$

and, according to Eq. (2.39), the function $\Phi(|x|)$ monotonically increases as $|x|$ increases.

Differentiating Eq. (2.46) with respect to time t along a solution of Eq. (2.32) yields

$$\dot{W} = -\psi(\dot{x})|\dot{x}| \leq 0 \tag{2.49}$$

and, hence, the total mechanical energy is a nonincreasing function of time. Let t_* be the instant of the first local extremum and $\bar{t} > t_*$ be any other instant of the local extremum of the function $x(t)$. Relationships $\dot{x}(t_*) = 0$, $\dot{x}(\bar{t}) = 0$, and $W(t_*) \geq W(\bar{t})$ imply that $\Phi(|x(t_*)|) \geq \Phi(|x(\bar{t})|)$ and, hence, due to monotonicity of the function $\Phi(|x|)$, $|x(t_*)| \geq |x(\bar{t})|$. This completes the proof of Proposition 2.1.

2.3.2.5 Proposition 2.2. The maximum absolute value of the acceleration in Eq. (2.34) is attained on the interval $0 \leq t \leq t_*$ where t_* is the instant of the first local extremum of the function $x(t)$.

2.3.2.6 Proof. According to Proposition 2.1, the inequality $|x(t_1)| \leq |x(t_*)|$ is valid for any $t_1 > t_*$. The function $x(t)$ is continuous, $x(0) = 0$, and t_* is the point of the absolute maximum of the function $|x(t)|$. Hence, for any instant $t_1 > t_*$, there exists $t_2 \leq t_*$ such that

$$|x(t_1)| = |x(t_2)|. \tag{2.50}$$

The relationship $W(t_1) \leq W(t_2)$, along with Eqs. (2.48) and (2.50), imply that

$$|\dot{x}(t_1)| \leq |\dot{x}(t_2)|. \tag{2.51}$$

For the solution to the initial-value problem of Eq. (2.32), we have $x(t) \geq 0, \dot{x}(t) \geq 0$ for $t \in [0, t_*]$, if $\beta > 0$, and $x(t) \leq 0, \dot{x}(t) \leq 0$ for $t \in [0, t_*]$, if $\beta < 0$. Therefore, $|\ddot{x}(t)| = \psi(\dot{x}(t)) + \varphi(x(t))$ for $t \in [0, t_*]$. This, together with Eqs. (2.37), (2.50), and (2.51), implies that

$$|\ddot{x}(t_1)| \leq \psi(\dot{x}(t_1)) + \varphi(x(t_1)) \leq \psi(\dot{x}(t_2)) + \varphi(x(t_2)) = |\ddot{x}(t_2)|. \tag{2.52}$$

Therefore, for any instant $t_1 > t_*$

$$|\ddot{x}(t_1)| \le |\ddot{x}(t_2)| \le \max_{t\in[0,t_*]} |\ddot{x}(t)|. \tag{2.53}$$

This completes the proof of Proposition 2.2.

According to Propositions 2.1 and 2.2, to calculate the performance criteria of Eqs. (2.33) and (2.34) it is sufficient to consider the behavior of the system of Eq. (2.32) only in the time interval $[0, t_*]$. Note that it may turn out that $t_* = \infty$. For instance, this is the case if $k = 0$ and $\psi(c, \dot{x}) = c\dot{x}^2$. In what follows, we assume $\beta > 0$ in Eq. (2.32). This does not lead to a loss of generality, because, as follows from Eq. (2.37), the change of the sign of variable x does not change Eq. (2.32). Under such an assumption, we have $x(t) \ge 0$ and $\dot{x}(t) \ge 0$ for $t \in [0, t_*]$ and, hence, the state trajectory of the system of Eq. (2.32) corresponding to its motion over the interval $[0, t_*]$ belongs to the first quadrant of the state plane.

Let us represent the initial-value problem of Eq. (2.32) on the time interval $[0, t_*]$ as the initial-value problem for the simultaneous first-order differential equations

$$\dot{x} = y, \quad \dot{y} = -\psi(c, y) - \varphi(k, x) \tag{2.54}$$

$$x(0) = 0, \quad y(0) = \beta, \quad t \in [0, t_*].$$

The solution $x(t), y(t)$ to the problem of Eq. (2.54) is represented graphically by a curve in the xy-plane. This curve is called the *state (phase) curve* or the *state (phase) trajectory*. The state trajectory of the system of Eq. (2.54) begins at the point $x = 0$ and $y = \beta$ at $t = 0$ and arrives at the point $x = I_1(c, k)$ and $y = 0$ at $t = t_*$. The functions $x(t)$ and $y(t), t \in [0, t_*]$, specify the state trajectory in parametric form, t being the parameter of the curve. Since $x(t)$ monotonically increases with t increasing from 0 to t_*, we can represent the state trajectory as a single-valued function $y = \widetilde{y}(x, c, k)$, $x \in [0, I_1(c, k)]$. Dividing the second equation in Eq. (2.54) by the first one, we arrive at the first-order differential equation for the state trajectory

$$\frac{dy}{dx} = -\frac{\psi(c, y) + \varphi(k, x)}{y}, \quad y(0) = \beta > 0. \tag{2.55}$$

It follows from Eq. (2.55) that $dy/dx < 0$ while $y > 0$, i.e., the function $\widetilde{y}(x, c, k)$ representing the state trajectory of the system in the first quadrant monotonically decreases with increasing x. Hence, this state trajectory can be represented by a single-valued function $x = \widetilde{x}(y, c, k)$ that is the inverse of $\widetilde{y}(x, c, k)$. The function $x = \widetilde{x}(y, c, k)$ satisfies the differential equation

$$\frac{dx}{dy} = -\frac{y}{\psi(c, y) + \varphi(k, x)}, \quad x(\beta) = 0, \quad y \in [0, \beta]. \tag{2.56}$$

2.3.2.7 Proposition 2.3. The maximum absolute value of the displacement of the body being isolated decreases with the growth of c and k, i.e.,

$$
\begin{aligned}
I_1(c_1,k) > I_1(c_2,k), &\quad \text{if} \quad c_1 < c_2; \\
I_1(c,k_1) > I_1(c,k_2), &\quad \text{if} \quad k_1 < k_2.
\end{aligned} \tag{2.57}
$$

2.3.2.8 Proof. We will prove the first of the inequalities of Eq. (2.57). Similar reasoning can be used to prove the second inequality. Introduce the function

$$
g(x,y) = y - \widetilde{y}(x,c_1,k). \tag{2.58}
$$

For all points lying in the first quadrant of the state plane below the state trajectory $y = \widetilde{y}(x,c_1,k)$, the inequality $g(x,y) < 0$ holds, whereas for all points above this trajectory, $g(x,y) > 0$. Differentiating the function $g(x,y)$ with respect to time along the solution to the system of Eq. (2.54) corresponding to $c_2 > c_1$ yields

$$
\dot{g} = \dot{y} - \frac{d\widetilde{y}}{dx}\dot{x} = -\left[\psi(c_2,y) + \varphi(k,x)\right] + \frac{\psi(c_1,\widetilde{y}(x,c_1,k)) + \varphi(k,x)}{\widetilde{y}(x,c_1,k)}y. \tag{2.59}
$$

Here, $x = x(t,c_2,k)$ and $y = y(t,c_2,k)$ is the solution to the initial-value problem of Eq. (2.54) for $c = c_2$ and a fixed k. It follows from Eqs. (2.59), (2.54), and (2.55) that $\dot{g} = \psi(c_1,\beta) - \psi(c_2,\beta)$ at $t = 0$. According to Eq. (2.41), the function $\psi(c,\beta)$ monotonically increases with respect to c, and we have $\dot{g} < 0$ for $t = 0$. Hence, because of the continuity of the function $\dot{g}$ in t, the inequality $\dot{g} < 0$ holds over some neighborhood of the initial instant $t = 0$. In this neighborhood, we have $\tilde{g}(t) = g(x(t,c_2,k), y(t,c_2,k)) < 0$, while $g(x(0,c_2,k), y(0,c_2,k)) = 0$. The function $\tilde{g}(t)$ does not vanish over the interval $0 < t \leq t_*$ and, hence, remains negative. Indeed, let $t' \in (0,t_*]$ be the first time instant at which the function $\tilde{g}(t)$ vanishes, i.e. $\tilde{g}(t') = 0$ while $\tilde{g}(t) < 0$ for all $t \in (0,t')$. This implies that $\dot{g} \geq 0$ for $t = t'$. However, $\tilde{g}(t') = 0$ entails $\dot{g} = \psi(c_1,\widetilde{y}(x,c_1,k)) - \psi(c_2,\widetilde{y}(x,c_1,k)) < 0$, in accordance with Eq. (2.41). This contradiction proves that for all $t \in (0,t_*]$ the state trajectory corresponding to $c_2 > c_1$ lies below the trajectory related to c_1 and therefore, $I_1(c_1,k) > I_1(c_2,k)$.

Before analyzing the properties of the performance criterion $I_2(c,k)$ we prove the following lemma.

2.3.2.9 Lemma 2.1. If functions $\psi(c,y)$ and $\varphi(k,x)$ satisfy the relationships of Eqs. (2.41) and (2.44), then the inequality

$$
\frac{d\varphi(k,\widetilde{x}(y,c,k))}{dk} > 0 \tag{2.60}
$$

holds for $y \in [0,\beta)$ and $k > 0$. The operator d/dk stands for differentiation of the composite function $\varphi(k,\widetilde{x}(y,c,k))$ with respect to k, with the other arguments being fixed.

2.3.2.10 Proof. Denote $S(k) = \widetilde{x}(\overline{y},c,k)$ for $\overline{y} \in [0,\beta)$ and $c \geq 0$. Multiply both sides of Eq. (2.55) by y and then integrate with respect to x from $x = 0$ to $x = S(k)$ to find

$$
\left(\overline{y}^2 - \beta^2\right)/2 + \int_0^{S(k)} \psi(c,\widetilde{y}(x,c,k))dx + \int_0^{S(k)} \varphi(k,x)dx = 0. \tag{2.61}
$$

Differentiate Eq. (2.61) with respect to k and then solve the resulting equation for $dS(k)/dk$ to obtain

$$\frac{dS(k)}{dk} = -\frac{\int_0^{S(k)} \left[\psi'_y(c, \widetilde{y}(x, c, k))\widetilde{y}'_k(x, c, k) + \varphi'_k(k, x)\right] dx}{\psi(c, \overline{y}) + \varphi(k, S(k))}. \tag{2.62}$$

Differentiate $\varphi(k, S(k))$ with respect to k and substitute Eq. (2.62) for $dS(k)/dk$ to obtain

$$\frac{d\varphi(k, S(k))}{dk} = \frac{\Phi_0(k, S(k)) + \Phi_1(k, S(k))}{\psi(c, \overline{y}) + \varphi(k, S(k))}, \tag{2.63}$$

where

$$\Phi_0(k, x) = \varphi'_k(k, x)\varphi(k, x) - \varphi'_x(k, x) \int_0^x \varphi'_k(k, \xi)d\xi, \tag{2.64}$$

$$\Phi_1(k, x) = \varphi'_k(k, x)\psi(c, \overline{y}) - \varphi'_x(k, x) \int_0^x \psi'_y(c, \widetilde{y}(\xi, c, k))\widetilde{y}'_k(\xi, c, k)d\xi. \tag{2.65}$$

According to Eq. (2.44), $\Phi_0(k, x) > 0$. For $\Phi_1(k, x)$, the inequality $\Phi_1(k, S(k)) \geq 0$ is valid. To prove this, it is sufficient to show that $\widetilde{y}'_k(x, c, k) < 0$ for all $x \in (0, S(k)]$.

Let $Y(x) = \widetilde{y}'_k(x, c, k)$. The function $Y(x)$ is governed by the initial-value problem

$$\frac{dY}{dx} = a(x)Y + b(x), \qquad Y(0) = 0, \tag{2.66}$$

where

$$a(x) = \frac{-\psi'_y(c, \widetilde{y}(x, c, k))\widetilde{y}(x, c, k) + \psi(c, \widetilde{y}(x, c, k)) + \varphi(k, x)}{[\widetilde{y}(x, c, k)]^2}, \tag{2.67}$$

$$b(x) = -\frac{\varphi'_k(k, x)}{\widetilde{y}(x, c, k)}. \tag{2.68}$$

Equation (2.66) results from differentiating Eq. (2.55) with respect to the parameter k. The solution to the problem of Eq. (2.66) is given by

$$Y(x) = \exp\left[a_1(x)\right] \int_0^x b(\xi) \exp\left[-a_1(\xi)\right] d\xi, \quad a_1(x) = \int_0^x a(\eta)d\eta. \tag{2.69}$$

It follows from Eq. (2.41) that $b(\xi) < 0$. Therefore, $Y(x) = \widetilde{y}'_k(x, c, k) < 0$ and, hence, $\Phi_1(k, x) \geq 0$. Equation (2.63) and the inequalities $\Phi_0(k, x) > 0$ and $\Phi_1(k, x) \geq 0$ imply

$$\frac{d\varphi(k, S(k))}{dk} = \frac{d\varphi(k, \widetilde{x}(\overline{y}, c, k))}{dk} > 0. \tag{2.70}$$

Since $\overline{y} \in [0, \beta)$ is arbitrary, this completes the proof of the lemma.

2.3.2.11 Proposition 2.4. The maximum force transmitted to the body being isolated does not decrease with the increase of the parameter k, i.e.,

$$I_2(c, k_2) \geq I_2(c, k_1) \quad \text{for} \quad k_2 > k_1. \tag{2.71}$$

In the region defined by inequality $I_2(c, k) > \psi(c, \beta)$, the function $I_2(c, k)$ monotonically increases with respect to k.

2.3.2.12 Proof. The performance criterion $I_2(c, k)$ can be represented as

$$I_2(c, k) = \max_{y \in [0, \beta]} \theta(y, c, k) \tag{2.72}$$

where

$$\theta(y, c, k) = \psi(c, y) + \varphi(k, \widetilde{x}(y, c, k)). \tag{2.73}$$

According to Lemma 2.1, the inequality $\theta(y, c, k_2) > \theta(y, c, k_1)$ holds for any $y \in [0, \beta)$, if $k_1 < k_2$, and, hence,

$$\max_{y \in [0, \beta]} \theta(y, c, k_2) \geq \max_{y \in [0, \beta]} \theta(y, c, k_1). \tag{2.74}$$

According to Eq. (2.74), the inequality of Eq. (2.71) holds. It follows from Eqs. (2.60), (2.72), and (2.73) that the equality in Eq. (2.71) occurs only if $I_2(c, k_1) = I_2(c, k_2) = \psi(c, \beta)$. This completes the proof of the proposition.

2.3.2.13 Proposition 2.5. Functions $I_1(c, k)$ and $I_2(c, k)$ do not reach the extremum values at internal points of their domains.

2.3.2.14 Proof. For the function $I_1(c, k)$, this proposition follows from Proposition 2.3, according to which the function $I_1(c, k)$ decreases with respect to either of the variables c or k. For the function $I_2(c, k)$, the proposition follows from Proposition 2.4. According to the latter proposition, the function $I_2(c, k)$ increases monotonically with respect to k, if $I_2(c, k) > \psi(c, \beta)$. The case $I_2(c, k) < \psi(c, \beta)$ cannot occur. Indeed, according to Eq. (2.54), with allowance for the fact that the functions $\psi(c, y)$ and $\varphi(k, x)$ are nonnegative (Assumption 1), we have

$$|\ddot{x}(0)| = |\dot{y}(0)| = \psi(c, \beta) + \varphi(k, 0) \tag{2.75}$$

It follows from Eqs. (2.75) and (2.34) that $I_2(c, k) \geq \psi(c, \beta)$. Consider now the case where $I_2(c, k) = \psi(c, \beta)$. Since $\beta \neq 0$, the function $\psi(c, \beta)$ monotonically increases with respect to c, according to Eq. (2.41). Thus, everywhere, the function $I_2(c, k)$ monotonically increases either with respect to the variable k or with respect to the variable c and, hence, cannot reach the extremum value at internal points of its domain.

Let us now establish the conditions under which the maximum load on the body being isolated is attained either at the initial time instant or at the instant $t = t_*$ of reaching the maximum displacement. In this case, the performance index $I_2(c, k)$ is given by

$$I_2(c,k) = \max_{y\in[0,\beta]} \theta(y,c,k) = \max\{\theta(0,c,k), \theta(\beta,c,k)\} \tag{2.76}$$

where $\theta(y, c, k)$ is the function defined by Eq. (2.73).

2.3.2.15 Proposition 2.6. Let functions $\psi(c, y)$ and $\varphi(k, x)$ satisfy Assumptions 1 through 5. Moreover, let them be twice continuously differentiable with respect to y and x for all $x \neq 0$ and $y \neq 0$, and let them satisfy the following inequalities:

$$\left(\frac{\partial^2\psi}{\partial y^2}y - \frac{\partial\psi}{\partial y}\right)\operatorname{sign}(y) \geq 0, \quad \frac{\partial^2\varphi}{\partial x^2} \geq 0. \tag{2.77}$$

Then the relation of Eq. (2.76) is valid.

2.3.2.16 Proof. First, perform some preliminary manipulations. Differentiate $\theta(y, c, k)$ with respect to y

$$\theta'_y = \psi'_y(c,y) + \varphi'_x(k, \widetilde{x}(y,c,k))\frac{d\widetilde{x}}{dy}. \tag{2.78}$$

According to Eqs. (2.56) and (2.73), $d\widetilde{x}/dy$ is expressed by

$$\frac{d\widetilde{x}}{dy} = -\frac{y}{\theta(y,c,k)}. \tag{2.79}$$

Substitute Eq. (2.79) into Eq. (2.78) to obtain

$$\theta'_y = \psi'_y(c,y) - \varphi'_x(k, \widetilde{x}(y,c,k))\frac{y}{\theta(y,c,k)}. \tag{2.80}$$

Differentiate Eq. (2.80) with respect to y and take into account Eq. (2.79)

$$\theta''_{yy} = \psi'_y + \varphi''_{xx}\left(\frac{y}{\theta}\right)^2 - \varphi'_x\frac{\theta - \theta'_y y}{\theta^2}. \tag{2.81}$$

Solve Eq. (2.80) for φ'_x , provided $y \neq 0$, to obtain

$$\varphi'_x = \frac{\psi'_y\theta - \theta'_y\theta}{y}. \tag{2.82}$$

Substitution of Eq. (2.82) into Eq. (2.81) yields

$$\theta''_{yy} = \psi''_{yy} - \frac{\psi'_y}{y} + \varphi''_{xx}\left(\frac{y}{\theta}\right)^2 + \frac{\psi'_y\theta'_y}{\theta} + \frac{\theta'_y}{y} - \frac{(\theta'_y)^2}{\theta}. \tag{2.83}$$

Now, let us prove the proposition. Since $\theta(y, c, k)$ is a continuous function of y, its maximum occurs for $y \in [0, \beta]$. Suppose that Eq. (2.76) is not true. Then, the global maximum of θ is attained at some internal point y_1 of the interval $[0, \beta]$, i.e. $y_1 \in (0, \beta)$. Since $\theta(y_1, c, k) > \theta(0, c, k)$, there exist $y_2 \in (0, y_1]$ and $\delta > 0$ such that $\theta'_y(y_2, c, k) = 0$ and $\theta'_y(y, c, k) > 0$ for $y \in [y_2 - \delta, y_2)$. Apply the mean value theorem to the function $\theta'_y(y, c, k)$ over the interval $[y_2 - \delta, y_2]$ to show that

$$\theta'_y(y_2, c, k) - \theta'_y(y_2 - \delta, c, k) = \theta''_{yy}(\xi, c, k)\delta, \tag{2.84}$$

where $\xi \in (y_2 - \delta, y_2)$. This implies $\theta''_{yy}(\xi, c, k) < 0$. Then, according to Eqs. (2.77) and (2.83), the inequality

$$\theta'_y(\xi, c, k)\left[\frac{\psi'_y(\xi, c)}{\theta(\xi, c, k)} + \frac{1}{\xi} - \frac{\theta'_y(\xi, c, k)}{\theta(\xi, c, k)}\right] < 0 \tag{2.85}$$

must hold. Since $\theta'_y(y_2, c, k) = 0$,we have $\theta'_y(\xi, c, k) \to 0$ as $\delta \to 0$, while the other quantities in Eq. (2.85) remain finite and positive. Therefore, by choosing δ sufficiently small, one can make the expression in square brackets positive. Thus, there exists $\xi \in (y_2 - \delta, y_2)$ such that $\theta'_y(\xi, c, k) < 0$. This contradicts the inequality $\theta'_y(y, c, k) > 0$ for $y \in [y_2 - \delta, y_2)$, and, hence, the maximum of θ cannot be attained at an internal point of the interval $0 \le y \le \beta$. This completes the proof of Proposition 2.6.

2.3.2.17 Summary of the Basic Properties of the Performance Indices. Let us summarize the basic properties of the performance criteria $I_1(c, k)$ and $I_2(c, k)$ that have just been established.

Under Assumptions 1 to 3, both the maximum absolute value of the displacement of the body being isolated, $I_1(c, k)$, and the maximum transmitted load, $I_2(c, k)$, occur on the time interval $[0, t_*]$, where t_* is the instant at which the relative velocity of the body vanishes for the first time (Propositions 2.1 and 2.2).

Under Assumptions 1 to 4, the maximum absolute value of the displacement of the body being isolated, $I_1(c, k)$, monotonically decreases as the damping and stiffness factors, c and k, increase (Proposition 2.3).

Under Assumptions 1 to 5, the maximum load transmitted to the body being isolated, $I_2(c, k)$, is a nondecreasing function of the stiffness factor k (Proposition 2.4). If, in addition, the inequalities of Eq. (2.77) hold, then the body experiences the maximum load either at the instant of shock ($t = 0$) or at the instant of occurrence of the maximum displacement (Proposition 2.6).

Under Assumptions 1 to 5, functions $I_1(c, k)$ and $I_2(c, k)$ do not reach the extremum values at internal points of their domains (Proposition 2.5). This implies that the extremum points of the functions $I_1(c, k)$ and $I_2(c, k)$ can lie only on the boundaries of the domains of these functions.

These properties which tend to be useful in their own right, considerably facilitate solving Problems 2.3 and 2.4 which involve choosing optimal parameters for isolator characteristics.

To conclude this section, we consider some examples of widely used passive isolators whose characteristics satisfy Assumptions 1 to 5 and the conditions of Eq. (2.77).

2.3.2.18 Example 2.1. *Isolators with Power Law Characteristics.* Consider isolators with power law characteristics given by

$$\psi(c,\dot{x}) = c\,|\dot{x}|^r\,, \quad \varphi(k,x) = k\,|x|^n\,, \\ r \geq 0, n \geq 0, c \geq 0, k \geq 0. \tag{2.86}$$

Such isolators are in rather widespread use. For instance, if $r = 1$ and $n = 1$, then we have the common linear isolator for which the characteristic of Eq. (2.30) becomes $u(x,\dot{x}) = c\dot{x} + kx$. It is readily verified that Assumptions 1 to 5 are satisfied by these isolators. If $r \geq 2$ and $n \geq 1$, functions $\psi(c,x)$ and $\varphi(k,x)$ given by (2.86) satisfy also the inequalities of Eq. (2.77).

2.3.2.19 Example 2.2. *An Isolator with Variable Stiffness.* As another example, consider the isolator with a variable stiffness characteristic expressed as

$$\varphi(k,x) = k\,|x|\,(1 + a\,|x|^n), \quad a \geq 0. \tag{2.87}$$

This characteristic satisfies Assumptions 1 to 4 for any $a \geq 0$ and $n \geq -1$. If, in addition, $n < (1+\sqrt{17})/2$, then Assumption 5 is also fulfilled. The inequality for $\varphi(k,x)$ in Eq. (2.77) is satisfied for $n \geq -1$.

2.3.2.20 Example 2.3. *Linear-Quadratic Damper.* Consider a damper consisting of linear and quadratic-law elements connected in series as shown in Fig. 2.2. The quadratic-law element 2 is attached to the body to be isolated (body m) and the linear element 1 is attached to the base. We assume that the masses of the damping elements 1 and 2 are negligibly small compared with the mass m of the body. Introduce the notation so that x is the displacement of body m relative to the base; x_1 is the displacement of the connection point C of elements 1 and 2 relative to the base; x_2 is the displacement of body m with respect to point C; c_1 is the damping coefficient of the linear damper; and c_2 is the damping coefficient of the quadratic-law damper.

According to the definition of Eq. (2.30) for the characteristic of an isolator consisting of stiffness and damping elements, the force applied to the body to be isolated by the damping element is

$$f_d = -\psi(c,\dot{x})\mathrm{sign}(\dot{x}). \tag{2.88}$$

Let us determine the function $\psi(c,\dot{x})$ for the damping device depicted in Fig. 2.2. The force f_d applied to body m is generated by the quadratic-law damper 2 and, accordingly, is given by

$$f_d = -c_2|\dot{x}_2|\dot{x}_2. \tag{2.89}$$

Since we have ignored the masses of the damping elements, the force applied by damper 2 at point C is equal in magnitude and opposite in sign to the force f_d of (2.89). Thus, we have

$$f_{2C} = -f_d = c_2|\dot{x}_2|\dot{x}_2. \tag{2.90}$$

The force applied to point C by the linear damper 1 is

$$f_{1C} = -c_1\dot{x}_1. \tag{2.91}$$

Since there is no mass concentrated at point C, in accordance with Newton's second law, we have

$$f_{1C} + f_{2C} = 0. \tag{2.92}$$

Substitute (2.90) and (2.91) into (2.92) to obtain

$$c_2|\dot{x}_2|\dot{x}_2 - c_1\dot{x}_1 = 0. \tag{2.93}$$

As is apparent from Fig. 2.2, the coordinates x_1, x_2, and x are related by

$$x = x_1 + x_2 \quad \text{or} \quad x_1 = x - x_2. \tag{2.94}$$

Substitute (2.94) into (2.93). This yields

$$|\dot{x}_2|\dot{x}_2 + 2\varkappa\dot{x}_2 - 2\varkappa\dot{x} = 0, \tag{2.95}$$

where

$$\varkappa = \frac{c_1}{2c_2}. \tag{2.96}$$

Solve Eq. (2.95) for x_2 to obtain

$$\dot{x}_2 = \begin{cases} -\varkappa + \sqrt{\varkappa^2 + 2\varkappa\dot{x}} & \text{for} \quad \dot{x} \geq 0 \\ \varkappa - \sqrt{\varkappa^2 - 2\varkappa\dot{x}} & \text{for} \quad \dot{x} < 0 \end{cases} \tag{2.97}$$

It is apparent from (2.97) that $\dot{x}_2 \geq 0$ for $\dot{x} \geq 0$ and $\dot{x}_2 < 0$ for $\dot{x} < 0$. By substituting (2.97) into (2.89) we obtain

$$f_d = -2\varkappa c_2 \left[|\dot{x}| + \varkappa - \sqrt{2\varkappa|\dot{x}| + \varkappa^2}\right] \operatorname{sign}(\dot{x}). \tag{2.98}$$

By comparison of (2.98) with (2.88),

$$\psi(c, \dot{x}) = c\left[|\dot{x}| + \varkappa - \sqrt{2\varkappa|\dot{x}| + \varkappa^2}\right], \tag{2.99}$$

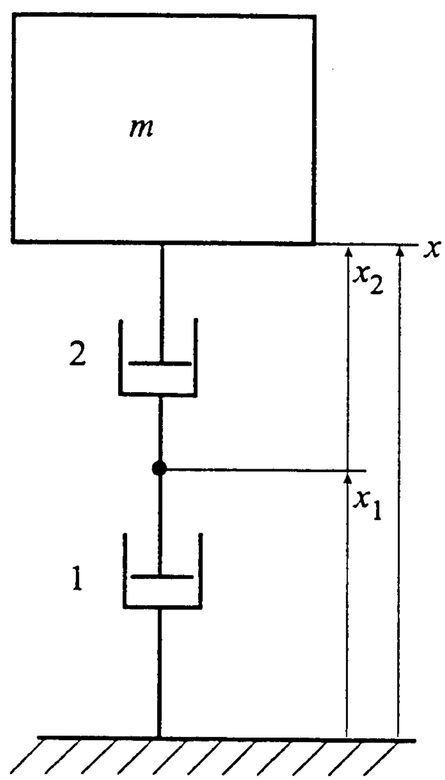


Figure 2-2. Linear and quadratic-law dampers connected in series .

1 - Linear damper ($c_1 \dot{x}_1$)

2 - Quadratic-law damper ($c_2 |\dot{x}_2| \dot{x}_2$)

$x = x_1 + x_2$ $\qquad$ $c_2 x_1 = c_2 |\dot{x}_2| \dot{x}_2$

where $c = 2\varkappa c_2 = c_1$.

It can be verified that the damper characteristic $\psi(c, \dot{x})$ of (2.99) satisfies Assumptions 1 to 4. However, it does not satisfy the first inequality in Eq. (2.77).

2.3.3 General Considerations Concerning the Calculation of the Optimal Parameters of the Isolator.

2.3.3.1 Solution of Problem 2.4. Problem 2.4 was stated in Section 2.3.1. See Eq. (2.36). In this section, methodology for solving this problem will be presented, utilizing the properties of the performance indices $I_1(c, k)$ and $I_2(c, k)$ established in the previous section.

As follows from Proposition 2.5, the minimum of the function $I_2(c, k)$ can be reached only on the boundary of the admissible region for the parameters c and k. Denote this region by Ω. According to Eq. (2.36),

$$\Omega = \{c, k : c \geq 0, \quad k \geq 0, \quad I_1(c, k) \leq D\}. \tag{2.100}$$

The boundary $\partial\Omega$ of this region can be defined as

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3, \tag{2.101}$$

where

$$\partial\Omega_1 = \{c, k : c \geq 0, k = 0, I_1(c, 0) \leq D\},$$

$$\partial\Omega_2 = \{c, k : c = 0, k \geq 0, I_1(0, k) \leq D\},$$

$$\partial\Omega_3 = \{c, k : c \geq 0, k \geq 0, I_1(c, k) = D\}.$$

The first set in this union is the portion of the nonnegative c-semiaxis belonging to Ω, the second set is the portion of the nonnegative k-semiaxis belonging to Ω, and the third set is the curve Γ_D defined by the equation

$$\Gamma_D = \{c, k : c \geq 0, k \geq 0, I_1(c, k) = D\}. \tag{2.102}$$

The region Ω and the components of its boundary are shown in Fig.2.3.

From the continuity properties of the solution of differential equations, it follows that the functions $I_1(c, k)$ and $I_2(c, k)$ are continuous. Since the function $I_1(c, k)$ is continuous and monotonically decreases with respect to either of the arguments c or k (Proposition 2.3), the curve Γ_D can be represented as a continuous monotonically decreasing function $c = c_D(k)$. This fact readily follows from the implicit function theorem.

The following propositions establish important facts concerning the location of the optimal parameters k^0 and c^0 of Problem 2.4 in the region Ω.

2.3.3.2 Proposition 2.7. The optimal parameters k^0 and c^0 lie on the curve Γ_D of Eq. (2.102).

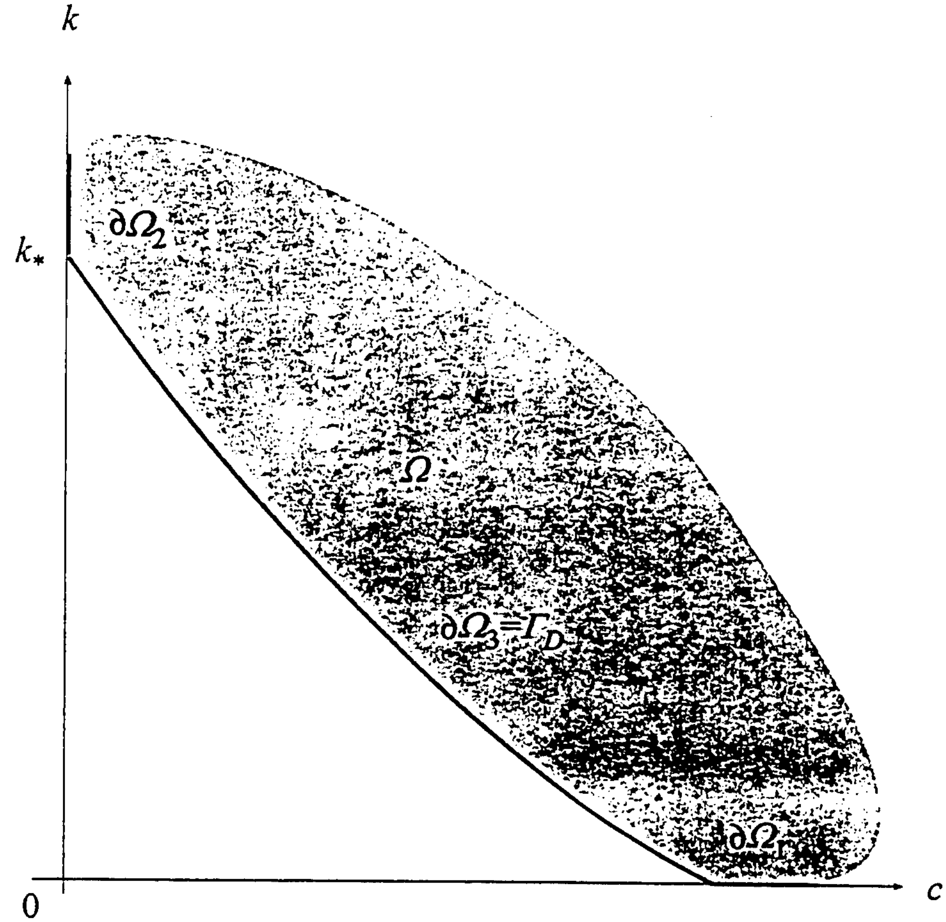


Figure 2-3. The region Ω and its boundary.

2.3.3.3 Proof. To prove this, we will show that the optimal parameters cannot lie on the portions of the c- and k-axes belonging to the admissible region Ω, except for the points of intersection of these axes with the curve Γ_D. Consider first the k-axis, on which $c = 0$. According to Eq. (2.42), we have $\psi(0, \dot{x}) = 0$. Hence, $I_2(0, k) > \psi(0, \beta) = 0$ for $k > 0$, and, according to Proposition 2.4, the function $I_2(0, k)$ monotonically increases with respect to k. Since the function $I_1(0, k)$ is continuous for $k > 0$, for any $k > 0$ satisfying the condition $I_1(0, k) < D$, one can find $k' < k$ such that $I_2(0, k') < I_2(0, k)$ and $I_1(0, k') < D$. Thus, the optimal solution cannot lie on the k-axis, except for the point of intersection of this axis with the curve Γ_D.

Consider now the c-axis, on which $k = 0$. In this case we have $I_2(c, 0) = \psi(c, \beta)$. This means that the force transmitted to the body being isolated assumes the maximum value at the initial instant, at which the velocity of the body assumes the maximum value. According to Eq. (2.41), the function $\psi(c, \beta)$ monotonically increases with respect to c. This, with allowance for the continuity of the function $I_1(c, 0)$, implies that the optimal solution cannot lie on the c-axis, except for the point of intersection of this axis with the curve Γ_D.

This completes the proof of the proposition.

2.3.3.4 Proposition 2.8. The curve Γ_D intersects the k-axis for any $D > 0$.

2.3.3.5 Proof. For $c = 0$, Eq. (2.32) has the energy integral.

$$W(x, \dot{x}) = \frac{\dot{x}^2}{2} + \int_0^{|x|} \varphi(k, \xi) d\xi = \frac{\beta^2}{2}. \tag{2.103}$$

The energy function $W(x, \dot{x})$ was introduced in Eqs. (2.46) and (2.48). The maximum displacement $I_1(c, k)$ is attained at the instant when the velocity $\dot{x}$ vanishes. At the point of intersection of the curve Γ_D with the k-axis (if such a point exists), we have $x = I_1(0, k) = D$. Substitute $x = D$ and $\dot{x} = 0$ into Eq. (2.103) to obtain the equation

$$\int_0^D \varphi(k, \xi) d\xi = \frac{\beta^2}{2} \tag{2.104}$$

for k. According to Eqs. (2.42) and (2.43), the left-hand side of Eq. (2.104) (1) is equal to zero for $k = 0$, (2) monotonically increases with respect to k, and (3) tends to infinity as $k \to \infty$. Hence, Eq. (2.104) has a unique solution $k = k_*$ which is the desired intersection point.

It follows from Proposition 2.8 that the optimal stiffness coefficient $k = k^0$ lies in the interval

$$0 \leq k^0 \leq k_* \tag{2.105}$$

It was indicated above that the curve Γ_D of Eq. (2.102) can be represented as a function $c = c_D(k)$, and, hence, the optimal stiffness k^0 and optimal damping c^0 are related by

$$c^0 = c_D(k^0). \tag{2.106}$$

Thus, the solution of Problem 2.4 can be reduced to the calculation of the minimum of the function

$$\hat{I}_2(k) = I_2(c_D(k), k) \tag{2.107}$$

of a single variable k on the interval $[0, k_*]$.

To calculate the function $\hat{I}(k)$ for a given k, one should execute the following steps:

Step 1. Solve the equation

$$I_1(c, k) = D \tag{2.108}$$

with respect to c to find $c = c_D(k)$. To calculate the function $I_1(c, k)$, integrate the initial value problem of Eq. (2.56) backward in the variable y to find

$$I_1(c, k) = \tilde{x}(0, c, k), \tag{2.109}$$

where $\tilde{x}(y, c, k)$ is the solution of the initial value problem of Eq. (2.56).

Step 2. For the value of $c = c_D(k)$ determined in the first step, calculate the maximum of the function

$$\hat{u}(y) = \psi[c_D(k), y] + \varphi[k, \tilde{x}(y, c_D(k), k))] \tag{2.110}$$

over the finite interval $[0, \beta]$ to find

$$\hat{I}_2(k) = \max_{y \in [0,\beta]} \hat{u}(y) \tag{2.111}$$

In the general case, steps 1 and 2, as well as the calculation of the minimum of the function $\hat{I}_2(k)$ of Eq. (2.107) are implemented numerically.

2.3.3.6 Solution of Problem 2.3. The solution procedure for Problem 2.3 (Eq. 2.35) could be presented similarly to that for Problem 2.4. For brevity, we omit this description but note that the Problem 2.3 is the reciprocal of Problem 2.4 in the sense of Theorem 1.1. Since the optimal parameters in Problem 2.4 lie on the curve of Eq. (2.102), we have $I_1(c^0, k^0) = D$. The optimal value $I_2(c^0, k^0)$ of the function $I_2(c, k)$ in Problem 2.4 decreases as D increases. This follows from the fact that the greater the D, the larger the admissible set Ω of Eq. (2.100). In turn, the global minimum of a function cannot increase if the domain in which the function is considered becomes larger. Thus, one can apply Theorem 1.1 to obtain the solution of Problem 2.3 from the solution of Problem 2.4.

2.3.4 Parametric Optimization of Isolators with Power Law Characteristics.

2.3.4.1 Preliminary Considerations. In this section, we consider the problem of the selection of an optimal isolator for an instantaneous shock from among the devices with power law characteristics of the form

$$u(x,\dot{x}) = c|\dot{x}|^r \mathrm{sign}(\dot{x}) + k|x|^n \mathrm{sign}(x), \quad r \geq 0, n \geq 0, c \geq 0, k \geq 0. \tag{2.112}$$

The characteristic of Eq. (2.112) is a particular case of Eq. (2.30) for $\psi(c,\dot{x}) = c|\dot{x}|^r$ and $\varphi(k,x) = k|x|^n$. If $r = 1$ and $n = 1$, then we have the common linear isolator for which $u(x,\dot{x}) = c\dot{x} + kx$. As mentioned in the previous section, Assumptions 1 to 5 are valid for such characteristics for any nonnegative c, k, r, and n, and in the case of $r \geq 2$ and $n \geq 1$, the inequalities of Eq. (2.77) are also satisfied. Therefore, the general propositions proved in Section 2.3.2 are applicable to the characteristics under consideration here.

Without loss of generality, we set $\beta = 1$ in Eq. (2.32), $U = 1$ in Eq. (2.35), and $D = 1$ in Eq. (2.36). This corresponds to using dimensionless (primed) variables and parameters given by

$$\begin{aligned} x' = \tfrac{xU}{\beta^2}\mathrm{sign}(\beta), \quad t' = \tfrac{Ut}{|\beta|}; \\ c' = \tfrac{c|\beta|^r}{U}, \quad k' = \tfrac{k\beta^{2n}}{U^{n+1}}, \end{aligned} \tag{2.113}$$

for Problem 2.3, and given by

$$\begin{aligned} x' = \tfrac{x}{D}\mathrm{sign}(\beta), \quad t' = \tfrac{|\beta| t}{D}; \\ c' = cD\,|\beta|^{r-2}, \quad k' = \tfrac{kD^{n+1}}{\beta^2}, \end{aligned} \tag{2.114}$$

for Problem 2.4.

Thus, the initial value-problem of Eq. (2.32) governing the relative motion of the body being isolated becomes

$$\begin{aligned} \ddot{x} + c|\dot{x}|^r \mathrm{sign}(\dot{x}) + k|x|^n \mathrm{sign}(x) = 0, \\ x(0) = 0, \quad \dot{x}(0) = 1, \end{aligned} \tag{2.115}$$

where primes are ignored.

Let us consider Problem 2.4, where the peak absolute acceleration of the body to be isolated (the peak force transmitted to the body by the isolator) is minimized, while the peak displacement of the body with respect to the base is constrained. We begin by analyzing the admissible region

$$\Omega = \{c, k : c \geq 0, k \geq 0, I_1(c,k) \leq 1\} \tag{2.116}$$

of parameters c and k satisfying the constraint $I_1(c,k) \leq 1$. The curve Γ_D of Eq. (2.102) for this region is given by

$$\Gamma_1 = \{c, k : c \geq 0, k \geq 0, I_1(c,k) = 1\}. \tag{2.117}$$

The curve Γ_1 of Eq. (2.117) is a portion of the boundary of the region Ω of Eq. (2.116). As was shown in Section 2.3.3, the curve of Eq. (2.117) can be represented as a monotone decreasing function $c = c_D(k) = c_1(k)$. The graph of this function is sketched in Fig. 2.3, where it is identified as $\partial\Omega_3$. The admissible region Ω is to the right of and above the curve $c = c_1(k)$.

It is of interest to analyze the behavior of the function $c_1(k)$ as $k \to 0$, depending on r. If $r < 2$, then $c_1(0)$ is finite and, hence, the curve Γ_1 of Eq. (2.117) intersects the c-axis, as shown in Fig. 2.4. To prove this, solve the initial-value problem of Eq. (2.56) for $\psi(c, y) = c|y|^r, \varphi(k, x) = k|x|^n$, and $k = 0$ to find

$$x = \bar{x}(y, c, 0) = \frac{1}{c}\int_y^1 \varsigma^{1-r} d\varsigma. \tag{2.118}$$

If $r < 2$, then the integral in Eq. (2.118) converges as $y \to 0$, and we have

$$\bar{x}(0, c, 0) = \frac{1}{c(2-r)}. \tag{2.119}$$

Substituting Eq. (2.119) into Eq. (2.109) yields

$$I_1(c, 0) = \frac{1}{c(2-r)}. \tag{2.120}$$

To find the point $c = c_*$ of the intersection of the curve Γ_1 of Eq. (2.117) with the c-axis, solve the equation $I_1(c, 0) = 1$ for c. Thus, we obtain

$$c_* = \frac{1}{2-r}. \tag{2.121}$$

If $r \geq 2$, then the integral in Eq. (2.118) diverges as $y \to 0$ and $c_1(k)$ tends to infinity as $k \to 0$. This situation is illustrated in Fig. 2.5.

2.3.4.2 Solution of Problem 2.4.

2.3.4.3 Case of $r \geq 2$ and $n \geq 1$. In this case, the inequalities of Eq. (2.77) are valid and, according to Proposition 2.6,

$$I_2(c, k) = \max\{c, k[I_1(c, k)]^n\}. \tag{2.122}$$

Consider on the ck-plane the curve B defined by

$$c = k[I_1(c, k)]^n. \tag{2.123}$$

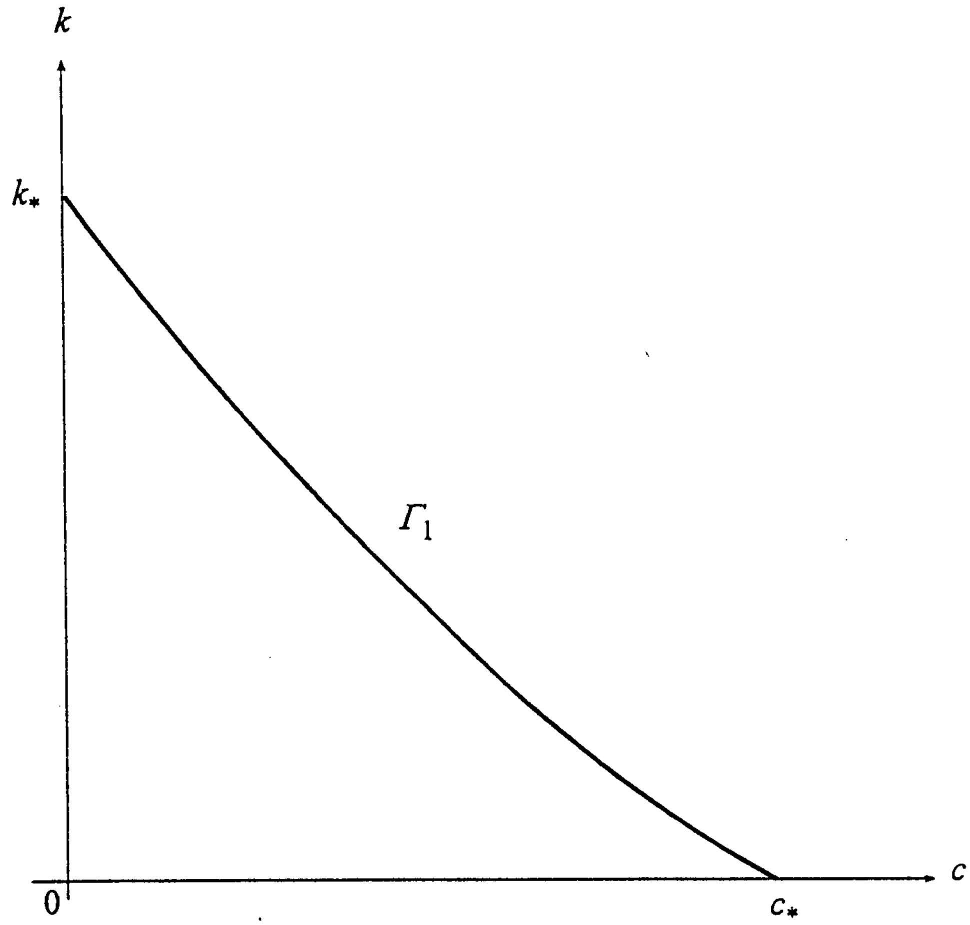


Figure 2-4. The curve Γ_1 for $r < 2$.

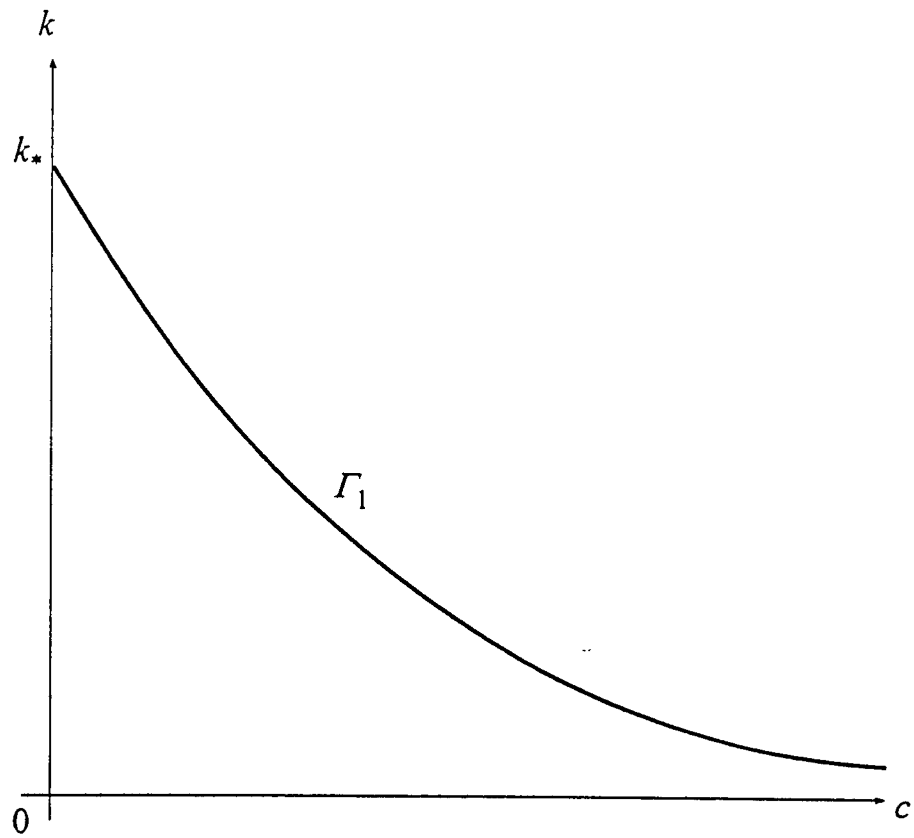


Figure 2-5. The curve Γ_1 for $r \geqslant 2$.

The right-hand side of Eq. (2.123) monotonically increases with respect to k and monotonically decreases with respect to c. The monotone decrease follows from the monotone decrease of the function $I_1(c, k)$ with respect to c, according to Proposition 2.3. To prove the monotone increase with respect to k, make use of expression $\varphi(k, x) = k|x|^n$ for the stiffness characteristic. Substitute $x = \tilde{x}(0, c, k)$ into this expression and use Eq. (2.109) to find $\varphi(k, \tilde{x}(0, c, k)) = k[I_1(c, k)]^n$. According to Lemma 2.1, the function $\varphi(k, \tilde{x}(0, c, k))$ monotonically increases with respect to k. It is then evident that the right-hand side of Eq. (2.123) monotonically increases with respect to k.

The implicit function theorem, with the properties just proved being taken into account, implies that Eq. (2.123) implicitly defines a monotonically increasing function $k = k_B(c)$. Moreover, the relation $k_B(0) = 0$ is valid. To prove this relation, note that for $c = 0$, there is no damping in the system of Eq. (2.115) and, hence, the energy of this system is conserved. Accordingly, we have

$$\frac{\dot{x}^2}{2} + \frac{k\,|x|^{n+1}}{n+1} = \frac{\dot{x}^2(0)}{2} = \frac{1}{2} \tag{2.124}$$

and hence,

$$I_1(0, k) = \max_{t \in [0, t_*]} |x(t)| = \left(\frac{n+1}{2k}\right)^{\frac{1}{n+1}}, \tag{2.125}$$

$$k\,[I_1(0, k)]^n = k^{\frac{1}{n+1}} \left(\frac{n+1}{2}\right)^{\frac{n}{n+1}}. \tag{2.126}$$

Therefore, the only solution of Eq. (2.123) for k with $c = 0$ is $k = 0$.

It now follows that the curve B divides the first quadrant of the ck-plane into two regions, B_1 and B_2, as shown in Fig. 2.6. Region B_1 lies below and to the right of the curve B, while B_2 lies above and to the left of this curve. For $(c, k) \in B_1$, $\quad I_2(c, k) = c$, whereas for $(c, k) \in B_2$, $\quad I_2(c, k) = k[I_1(c, k)]^n$.

Denote by R the intersection point of the curves Γ_1 of Eq. (2.117) and B (Fig. 2.7). The values of parameters c and k corresponding to the point R are the desired optimal parameters of the isolator in the case of $r \geq 2$ and $n \geq 1$. This can be shown as follows. According to Proposition 2.7, the optimal parameters lie on the curve Γ_1. According to Eq. (2.122) and the relation $I_1(c, k) = 1$, which is satisfied on the curve Γ_1, on this curve we have $I_2(c, k) = c$ to the right of the point R and $I_2(c, k) = k[I_1(c, k)]^n = k$ to the left of the point R. See Fig. 2.6. Thus, the function $I_2(c, k)$ decreases when approaching the point R from both sides along the curve Γ_1. Hence, the point R is the point of minimum of the criterion $I_2(c, k)$.

The coordinates $c = c^0$ and $k = k^0$ of the point R are the optimal parameters in Problem 2.4 for the case of $r \geq 2$, $n \geq 1$. The optimal parameters are determined uniquely.

Analytically, the optimal parameters satisfy the following simultaneous equations:

$$I_1(c, k) = 1, \quad c = k[I_1(c, k)]^n. \tag{2.127}$$

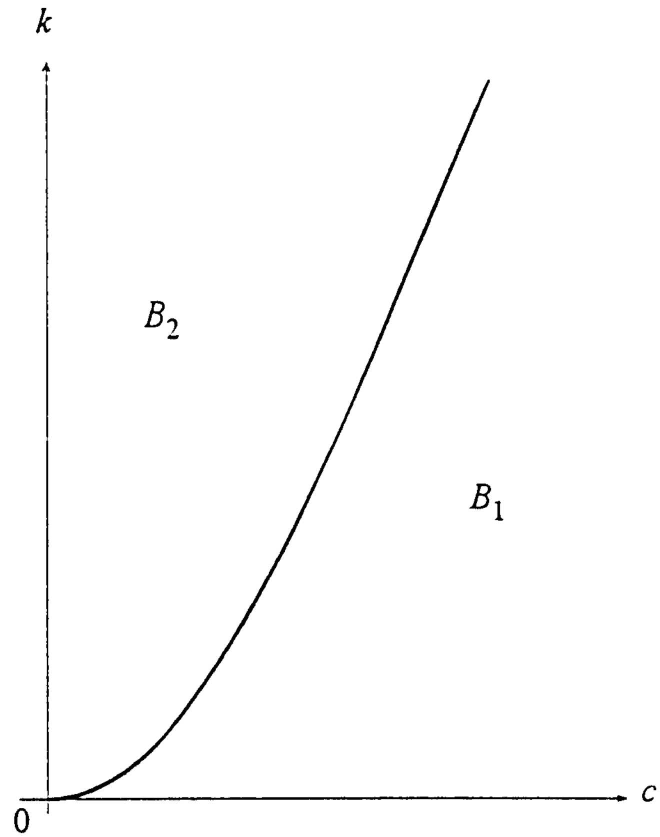


Figure 2-6. Regions B_1 and B_2 and their boundary.

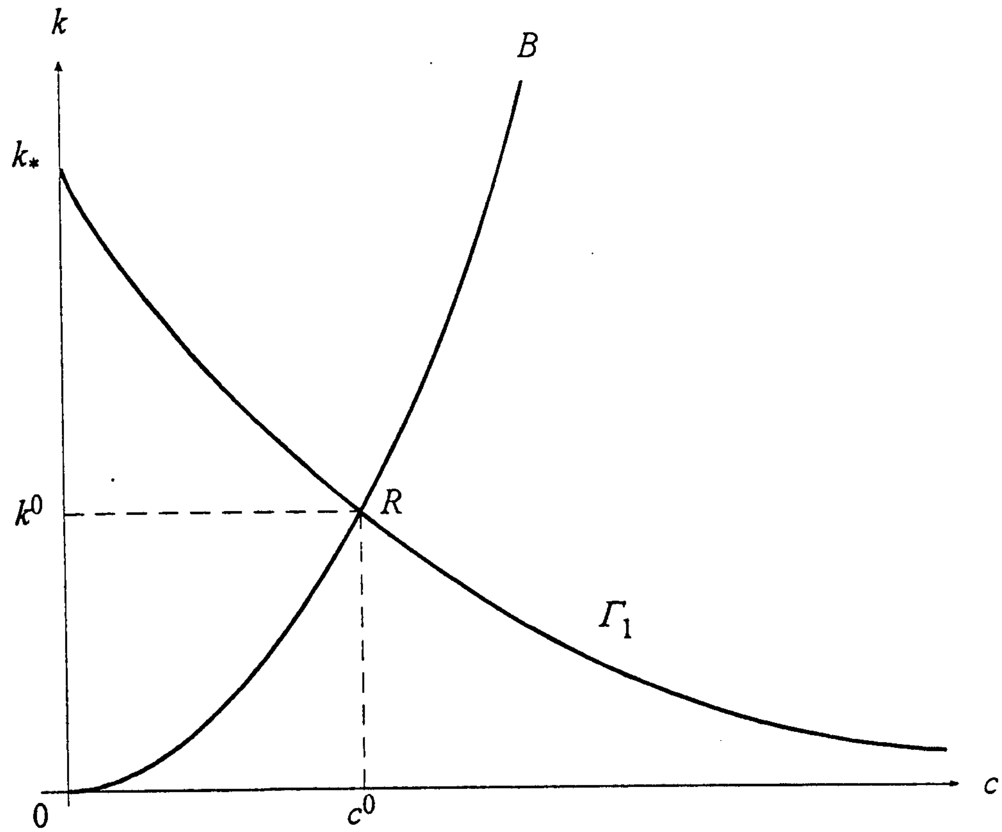


Figure 2-7. Graphic determination of the optimal parameters in Problem 2.4 for $r \geqslant 2$ and $n \geqslant 1$.

Since $I_1(c,k) = 1$ at the point $R, c = k$ at this point and, hence, the optimal parameters c^0 and k^0 are determined as

$$I_1(k^0, k^0) = 1, \quad c^0 = k^0. \tag{2.128}$$

According to Eq. (2.128), to find the optimal parameters only the nonlinear algebraic equation

$$I_1(k,k) = 1 \tag{2.129}$$

needs to be solved.

It is worth noting that to find the optimal parameters minimizing the performance criterion $I_2(c,k)$ it is not necessary to calculate this criterion.

Calculation of the criterion $I_1(c,k)$ requires integration of Eq. (2.115), which in the general case is not amenable to analytical methods. Rather, a numerical scheme needs to be used. Therefore, Eq. (2.129) for the optimal k is generally solved numerically. The function $I_1(k,k)$ is a continuous, monotonically decreasing function of k, $I_1(k,k) \to \infty$ as $k \to 0$, and $I_1(k,k) \to 0$ as $k \to \infty$. Hence, there exists a unique positive root of Eq. (2.129). To find this root, one may use numerical methods for determining the root of a function of a single variable. In our opinion, the most suitable methods in this situation are those that do not require the calculation of derivatives, e.g., the bisection method.

2.3.4.4 The Other Cases. The method of calculation of the optimal parameters in the case of $r \geq 2$ and $n \geq 1$ was based on the relation of Eq. (2.122). In the other cases, this relationship does not, in general, hold. Therefore, in these cases, one has to use the general technique described in Section 2.3.3. According to this technique, the search for the optimal parameters is reduced to the minimization of the function $\hat{I}_2(k)$ of Eq. (2.107) on the interval $[0, k_*]$. In the case in question, we have $\hat{I}_2(k) = I_2(c_1(k), k)$. The point k_* of the intersection of the curve Γ_1 with the k-axis is expressed as $k_* = (n+1)/2$. This expression can be obtained by setting the right-hand side of Eq. (2.125) equal to unity and solving the resulting equation for k. Thus, the optimal parameters c^0 and k^0 are determined according to the relations

$$I_2(c_1(k^0), k^0) = \min_{k \in [0,(n+1)/2]} I_2(c_1(k), k), \quad c^0 = c_1(k^0). \tag{2.130}$$

Note that the number of computations needed to calculate the optimal parameters in the general case is substantially larger than that needed to calculate the optimal parameters in the case of $r \geq 2$ and $n \geq 1$. In the latter case, the calculation of optimal parameters is reduced to the solution of only one equation of Eq. (2.129). No minimization is required. Moreover, one does not even need to calculate the criterion $I_2(c,k)$ to be minimized. The determination of the optimal parameters according to Eq. (2.130) in the general case requires the calculation of both functions $I_1(c,k)$ and $I_2(c,k)$ and involves the minimization with respect to k.

2.3.4.5 Optimal Parameters. The optimal parameters corresponding to the solution of Problem 2.4, in the dimensionless variables of Eq. (2.114), are determined from Eq. (2.128) if $r \geq 2$ and

$n \geq 1$ and from Eq. (2.130) if $r < 2$ or $n < 1$. Returning to dimensional variables according to Eq. (2.114) we can express the dependence of the optimal damping and stiffness factors on the initial velocity β imparted by the shock to the body being isolated and on the maximum allowable displacement D:

$$c^0 = \frac{(c^0)'}{D|\beta|^{r-2}}, \quad k^0 = (k^0)'\frac{\beta^2}{D^{n+1}}. \tag{2.131}$$

Here $(c^0)'$ and $(k^0)'$ denote dimensionless optimal parameters, while c^0 and k^0 are the corresponding dimensional values for the optimal parameters. The values of the performance criteria at the optimal parameters are given by

$$I_2(c^0, k^0) = I_2'[(c^0)', (k^0)']\frac{\beta^2}{D}, \quad I_1(c^0, k^0) = D, \tag{2.132}$$

where $I_2'[(c^0)', (k^0)']$ is the dimensionless minimum value of the force transmitted to the body. Note that dimensionless optimal values $(c^0)', (k^0)'$, and $I_2'[(c^0)', (k^0)']$ depend neither on β nor on D. They depend only on the exponents r and n of the damper and stiffness elements.

This completes the solution of Problem 2.4.

2.3.4.6 Solution of Problem 2.3. The solution to Problem 2.3, in which the peak displacement of the body to be isolated relative to the base is minimized and the maximum force transmitted to the body is constrained, can be obtained with the help of Theorem 1.1 that is proved in Section 1.2. According to Eqs. (2.35), (2.36), and (2.132), Problems 2.3 and 2.4 are reciprocal to each other and therefore, the solution to Problem 2.3 is given by

$$c_0 = \frac{(c^0)'}{I_2'[(c^0)', (k^0)']}\frac{U}{|\beta|^r}, \quad k_0 = \frac{(k^0)'}{\{I_2'[(c^0)', (k^0)']\}^{n+1}}\frac{U^{n+1}}{\beta^{2n}}; \tag{2.133}$$

$$I_1(c_0, k_0) = I_2'[(c^0)', (k^0)']\frac{\beta^2}{U}, \quad I_2(c_0, k_0) = U. \tag{2.134}$$

2.3.4.7 Comparison of Power Law Isolators with Different Exponents. To conclude this section, we present the results of the solution of Problem 2.4 obtained according to the technique described above for different combinations of exponents $r = 1, 2, 3, 4$ and $n = 1, 2, 3, 4$. The numerical results of the solution are brought together in Table 2.1. In this table, for brevity, the notation I_2' is used instead of $I_2'[(c^0)', (k^0)']$. Note that the problem admits a complete analytical solution in the case of $r = 2$ and $n = 1$, and the corresponding data in the table are exact. This analytical solution is presented in the following section where the problem of determining power law characteristics that ensure the limiting protection against shock loadings is considered. For combinations of parameters other then $r = 2$ and $n = 1$, the results, displayed in Table 2.1, were obtained numerically.

It is remarkable that the isolator with a quadratic damper ($r = 2$) and a linear spring ($n = 1$) gives $I_2'[(c^0)', (k^0)'] = 0.5$, thereby implementing the limiting capabilities of protection against the

impulse loading $F(t) = \beta\delta(t)$, evaluated by Eq. (2.26) or (2.28). Very good shock isolation is also provided by the linear isolator ($r = 1, n = 1$), with the optimal parameters. For such an isolator, $I_2'[(c^0)', (k^0)'] = 0.521$, which is only 4% higher than the absolute minimum. Generally, the tendency of an increase in the minimum value of the optimization criterion as r and n increase is apparent in Table 2-1.

Table 2-1. Optimal values of the stiffness and damping coefficients and the minimum of the peak acceleration for isolators with power law characteristics.

$r \backslash n$		1	2	3	4
1	I_2'	0.521	0.528	0.560	0.588
	$(c^0)'$	0.485	0.528	0.560	0.588
	$(k^0)'$	0.361	0.441	0.497	0.539
2	I_2'	0.500	0.597	0.672	0.729
	$(c^0)'$	0.500	0.597	0.672	0.729
	$(k^0)'$	0.500	0.597	0.672	0.729
3	I_2'	0.557	0.685	0.785	0.873
	$(c^0)'$	0.557	0.685	0.785	0.873
	$(k^0)'$	0.557	0.685	0.785	0.873
4	I_2'	0.602	0.756	0.885	0.992
	$(c^0)'$	0.602	0.756	0.885	0.992
	$(k^0)'$	0.602	0.756	0.885	0.992

2.3.5 Power Law Characteristics Implementing the Limiting Capabilities of Shock Protection.

Consider now the determination of the values of parameters c, k, r, and n for damping and stiffness characteristics given by Eq. (2.112) that ensure that the limiting shock protection quality of Section 2.2 is achieved. (Recall that we are considering the protection from an instantaneous impact characterized by the initial velocity $\beta \neq 0$ of the body to be isolated. By introducing dimensionless variables, one can always make $\beta = 1$.) As shown in Section 2.2, for the case where the performance index to be minimized is the maximum force transmitted to the body being isolated and the constraint is imposed on its maximum displacement, the optimal control $u^0(t)$ in dimensionless variables of Eq. (2.16) is given by Eq. (2.25) and is determined uniquely for the interval $0 \leq t < 2$. The corresponding motion of the body is described by Eq. (2.27). Hence, for the desired values of parameters c, k, r, and n, Eq. (2.115), with the expressions of Eq. (2.27) being substituted for x and $\dot{x}$, must be satisfied identically on the interval $0 \leq t \leq 2$. i.e.,

$$c(1 - \frac{t}{2})^r + k(t - \frac{t^2}{4})^n = \frac{1}{2}, \qquad 0 \leq t \leq 2. \tag{2.135}$$

Consider first the case where $r > 0$ and $n > 0$. Substitution of $t = 0$ and $t = 2$ into Eq. (2.135) shows that the damping and stiffness coefficients must be equal to $0.5 : c^0 = k^0 = 0.5$. Simple manipulations show that the identity of Eq. (2.135) is fulfilled if $c = k = 0.5, n = 1$. and $r = 2$. No other positive values of the exponents n and r satisfy this identity. Indeed, substituting $t = 1$ and

$t = 3/2$ into Eq. (2.135), with $c = k = 0.5$, we obtain $2^r = 4^n/(4^n - 3^n)$ and $2^r = 4^n/(16^n - 15^n)^{1/2}$. These equalities imply the relation $4^n - 3^n = (16^n - 15^n)^{1/2}$. Squaring this equation and then dividing by 12^n we arrive at the equivalent equation

$$g(n) = 2, \quad g(n) = (3/4)^n + (5/4)^n. \tag{2.136}$$

It can be verified that

$$g(0) = g(1) = 2 \tag{2.137}$$

$$\frac{d^2g(n)}{dn^2} = \left(\frac{3}{4}\right)^n \left(\ln\frac{3}{4}\right)^2 + \left(\frac{5}{4}\right)^n \left(\ln\frac{5}{4}\right)^2 > 0. \tag{2.138}$$

This means that $n = 0$ and $n = 1$ are roots of the equation in question, while the function $g(n)$ is convex and its graph has no more than two common points with the straight line $g = 2$. This implies that the equation $g(n) = 2$ and, hence, the equation $4^n - 3^n = (16^n - 15^n)^{1/2}$, have only a single positive root $n = 1$. To determine the parameter r substitute $n = 1$ into the equation $2^r = 4^n/(4^n - 3^n)$ or $2^r = 4^n/(16^n - 15^n)^{1/2}$ and solve this equation for r. This yields $r = 2$.

From this and from Propositions 2.1 and 2.2, it follows that the isolator with a linear spring and a quadratic damper implements the limiting shock isolation capabilities when the damping and stiffness factors are given by $c^0 = k^0 = 0.5$. No other isolator with power law characteristics possesses such a property for positive exponents r and n.

It follows from Eq. (2.135) that, in addition to the isolator with a linear spring and a quadratic damper, the limiting isolation capabilities are also provided by isolators whose parameters are given by

a) $c = 0, k = 0.5, n = 0$;

b) $c = 0.5, k = 0, r = 0$;

c) c and k are any positive numbers satisfying the condition $c + k = 0.5$, while $n = 0$ and $r = 0$.

Case (a) corresponds to the undamped isolator with a bang-bang spring ($u(x, \dot{x}) = 0.5\,\mathrm{sign}(x)$). Case (b) corresponds to the dry friction damper with no elastic element ($u(x, \dot{x}) = 0.5\,\mathrm{sign}(\dot{x})$). Finally, case (c) corresponds to the bang-bang spring and dry friction damper connected in parallel, the sum of the stiffness and friction coefficients being equal to 0.5 ($u(x, \dot{x}) = c\,\mathrm{sign}(\dot{x}) + k\,\mathrm{sign}(x)$, $c + k = 0.5$).

The four types of isolators described above cover all passive isolators with power law characteristics defined by Eq. (2.112) that provide the limiting capabilities of protection against the impulse loading $F = \beta\delta(t)$. The limiting capabilities are characterized by the value $I_2(c^0, k^0) = 0.5$ for the peak force transmitted to the body being isolated, provided that $\beta = 1$ and the displacement is constrained by $I_1(c, k) \leq 1$. The peak displacement of the body to be isolated, with the optimal parameters, assumes its maximum allowable value, i.e. $I_1(c^0, k^0) = 1$.

The expressions presented above are related to the case of $\beta = D = 1$ that corresponds to using dimensionless variables and parameters according to Eqs. (2.16) or (2.114). Returning to the initial dimensional variables we obtain the dependence of the optimal parameters and corresponding values of performance criteria on the relative velocity β imparted by the shock to the body being isolated and the maximum allowable relative displacement D.

2.3.5.1 Basic Results. We will summarize the results of Section 2.3.5 in dimensional variables. Among passive isolators with power law characteristics defined by Eq. (2.112), the limiting isolation capabilities are provided by the isolators (and only by them) enumerated next.

1) The isolator with a linear spring ($n = 1$) and a quadratic damper ($r = 2$), with the stiffness and damping factors given by

$$c = c^0 = \frac{1}{2D}, \quad k = k^0 = \frac{\beta^2}{2D^2}; \tag{2.139}$$

2) The undamped ($c = c^0 = 0$) isolator with a bang-bang spring ($n = 0$) the stiffness factor of which is given by

$$k = k^0 = \frac{\beta^2}{2D}; \tag{2.140}$$

3) The isolator consisting of a dry friction damper ($r = 0$) with no stiffness element ($k = k^0 = 0$), the friction coefficient being

$$c = c^0 = \frac{\beta^2}{2D}; \tag{2.141}$$

4) The isolator with a bang-bang spring ($n = 0$) and a dry friction damper ($r = 0$), the stiffness and friction coefficients being related by

$$c^0 + k^0 = \frac{\beta^2}{2D}, \quad c^0 > 0, \quad k^0 > 0. \tag{2.142}$$

For all the isolators, the peak load and the maximum relative displacements are expressed as

$$I_2(c^0, k^0) = \frac{\beta^2}{2D}, \quad I_1(c^0, k^0) = D. \tag{2.143}$$

The relations of Eq. (2.143) show that with optimal parameters c^0 and k^0, the maximum relative displacement of the body being isolated is equal to the maximum allowable value D, while the value of the functional to be minimized (the peak load $I_2(c, k)$) monotonically decreases with D increasing. Hence, using Theorem 1.1 one can find optimal parameters c_0 and k_0 providing the absolute minimum of the peak displacement of the body being isolated when the peak

acceleration, which is proportional to the maximum force transmitted to the body, is constrained by a constant U. We present the final result omitting the conventional procedure of recalculating optimal parameters taking advantage of the duality of the two problems. Among passive isolators with power law characteristics defined by Eq. (2.112), the maximum relative displacement of the body being isolated is provided by the isolators (and only by them) enumerated next.

1) The isolator with a linear spring ($n = 1$) and a quadratic damper ($r = 2$), with the stiffness and damping factors given by

$$c = c_0 = \frac{U}{\beta^2}, \quad k = k_0 = \frac{2U^2}{\beta^2}; \tag{2.144}$$

2) The undamped ($c = c_0 = 0$) isolator with a bang-bang spring ($n = 0$), the stiffness factor of which is

$$k = k_0 = U; \tag{2.145}$$

3) The isolator consisting of a dry friction damper ($r = 0$) and no stiffness element ($k = k_0 = 0$), the friction coefficient being

$$c = c_0 = U; \tag{2.146}$$

4) The isolator with a bang-bang spring ($n = 0$) and a dry friction damper ($r = 0$), the stiffness and friction coefficients of which are related by

$$c_0 + k_0 = U, \qquad c_0 > 0, \quad k_0 > 0. \tag{2.147}$$

In all four cases the performance criteria $I_1(c, k)$ and $I_2(c, k)$ assume the values

$$I_1(c_0, k_0) = \frac{\beta^2}{2U}, \quad I_2(c_0, k_0) = U. \tag{2.148}$$

Of the four types of isolators that have power law characteristics and implement the limiting shock protection capabilities, the most convenient for practical applications is the isolator with a linear spring and a quadratic damper. A body being isolated, when connected to a base by such an isolator, has a single equilibrium position $x = 0$ with respect to the base and, moreover, this equilibrium position is asymptotically stable. Due to this, the body returns to the initial position after oscillations caused by the shock have decayed. The other three types of isolators do not possess such a property. The undamped isolator with a bang-bang spring has no dissipation, and oscillations of the body do not decay. The system with a dry friction damper has a sticking zone and the return to the initial position is not guaranteed. The guaranteed return of the body being isolated to the initial position, which occurs for the isolator with a linear spring and a quadratic

damper, makes such isolators applicable for the protection against repeated shocks separated by time intervals sufficient for the oscillations to decay.

2.3.6 Two-Criteria Optimization. Pareto-Optimal Sets.

In Section 1.2.4 of Chapter 1 we outlined the approach for treating multicriteria optimization problems by the construction of Pareto-optimal sets and found the Pareto-optimal set for parameters of a linear spring-damper shock isolator. As performance criteria, the quantities $I_1(c,k)$ and $I_2(c,k)$ defined by Eqs. (2.33) and (2.34) were chosen. In this section, we present some additional results concerning Pareto-optimal sets associated with optimization of stiffness and damping factors for shock isolators with power law characteristics. The performance criteria are the peak displacement of the body being isolated with respect to the base ($I_1(c,k)$) and the peak load transmitted to the body ($I_2(c,k)$).

Equations (2.131) and (2.133) can be regarded as equations of curves defined parametrically in the ck-plane. The parameter of the curve of Eq. (2.131) is D, while the parameter of the curve of Eq. (2.133) is U. Eliminating D and U from the corresponding equations we express k^0 and k_0 through c^0 and c_0, respectively, as

$$k^0 = \lambda (c^0)^{n+1}, \quad k_0 = \lambda (c_0)^{n+1}, \qquad (2.149)$$

$$\lambda = \frac{(k^0)' \, |\beta|^{r(n+1)-2n}}{[(c^0)']^{n+1}}$$

Recall that c_0 and k_0 are the optimal values of the damping and stiffness coefficients, respectively, in Problem 2.3 (Eq. (2.35)). The optimal coefficients depend on the input parameters of the problem, in particular, on the shock intensity β and the maximum acceleration U allowable for the body being isolated. Similarly, c^0 and k^0 are the optimal damping and stiffness coefficients in Problem 2.4 (Eq. (2.36)). The quantities c^0 and k^0 depend on the shock intensity β and the maximum displacement D allowable for the body. The quantities $(c^0)'$ and $(k^0)'$ are the optimal damping and stiffness coefficients in Problem 2.4, when represented in the dimensionless variables introduced in Eq. (2.114). They depend only on the exponents n and r of the damping and stiffness characteristics.

It follows from Eq. (2.149) that Eqs. (2.131) and (2.134) define in the ck-plane the same curve

$$k = \lambda c^{n+1}, \quad \lambda = \frac{(k^0)' \, |\beta|^{r(n+1)-2n}}{[(c^0)']^{n+1}}. \qquad (2.150)$$

Note that the curve of Eq. (2.150) is unique if dimensionless optimal parameters $(k^0)'$ and $(c^0)'$ resulting from Eq. (2.128) for $r \geq 2$ and $n \geq 1$ or from Eq. (2.130) in the other cases are determined uniquely. It was proved in Section 2.3.4 that $(k^0)'$ and $(c^0)'$ are obtained uniquely in the case of $r \geq 2$ and $n \geq 1$. The dimensionless optimal parameters are also obtained uniquely for some other isolators with power law characteristics. In particular, numerical analysis shows that the optimal solution is unique for the isolators with $r = 1$ and $n = 1, 2, 3, 4$. However, this is not in

general the case for isolators with power law characteristics. For example, if $r = 0$ and $n = 0$ in Eq. (2.112) then, as shown in Section 2.3.4, any nonnegative $(k^0)'$ and $(c^0)'$ related by $(k^0)' + (c^0)' = 0.5$ are optimal parameters. In such cases, instead of one curve, we have a bundle of curves of Eq. (2.150), different curves corresponding to different λ.

Let us denote the curve of Eq. (2.150) by Π_λ. Any point of the curve Π_λ is a Pareto-optimal point. This follows from the fact that any point of the curve Π_λ gives optimal parameters in Problem 2.4 for some D and $I_1(c, k) = D$ for the optimal parameters (the latter was proved in Section 2.3.4). Suppose that a point $(c, k) \in \Pi_\lambda$ is not a Pareto-optimal point. Then there exists another point $(\bar{c}, \bar{k})$ such that either $I_1(\bar{c}, \bar{k}) < I_1(c, k)$ and $I_2(\bar{c}, \bar{k}) \leq I_2(c, k)$ or $I_1(\bar{c}, \bar{k}) \leq I_1(c, k)$ and $I_2(\bar{c}, \bar{k}) < I_2(c, k)$. The first of these possibilities contradicts the fact that the optimal parameters in Problem 2.4 always lie on the curve $I_1(c, k) = D$. The second possibility contradicts the assumption that the point (c, k) corresponds to optimal parameters in Problem 2.4 for some D. Hence, the point (c, k) is Pareto-optimal.

Now, let us prove that any Pareto-optimal point must belong to a curve of Eq. (2.150). Indeed, if a point (c^*, k^*) does not belong to Π_λ, then this point does not correspond to optimal parameters in Problem 2.4 with $D = I_1(c^*, k^*)$. Therefore, one can find c and k such that $I_2(c, k) < I_2(c^*, k^*)$ while $I_1(c, k) \leq I_1(c^*, k^*)$ and hence, the point (c^*, k^*) is not Pareto-optimal.

Thus, we have proved that the Pareto-optimal set Π in the parameter plane (ck-plane) for isolators with power law characteristics is a union (over λ) of curves Π_λ

$$\Pi = \bigcup_{\lambda \in \Lambda} \Pi_\lambda \tag{2.151}$$

where Λ is the set of possible λ in Eq. (2.150). The number of elements of the Λ set depends on the number of different solutions to Problem 2.4 at a fixed D (or to Problem 2.3 at a fixed U). As mentioned above, the solution is unique for all characteristics with $r \geq 2$ and $n \geq 1$ and for a number of other characteristics, in particular, for the characteristics with $r = 1$ and $n = 1, 2, 3, 4$. In such cases, the set Λ consists of only one point and the Pareto-optimal set is a curve. This situation is shown in Fig. 2.8. In contrast, in the case of $r = 0$ and $n = 0$, corresponding to a dry-friction damper and a bang-bang spring, Problem 2.4 has a continuum of solutions. Any nonnegative $(k^0)'$ and $(c^0)'$ related by $(k^0)' + (c^0)' = 0.5$ can be substituted into the expressions of Eq. (2.150) for λ, and, depending on the particular values of the parameters, λ can vary from 0 (for $(k^0)' = 0, (c^0)' = 0.5$) to $+\infty$ (for $(k^0)' = 0.5, (c^0)' = 0$). Hence, in this case, $\Lambda = [0, \infty)$ and, according to Eqs. (2.150) and (2.151), the Pareto-optimal set is the whole of the first quadrant of the ck-plane.

In conclusion, some remarks may be appropriate concerning the Pareto-optimal set in the performance criteria plane ($I_1 I_2$-plane). We denote such a set by Π^*. This set is defined as the mapping of the Π set onto the $I_1 I_2$-plane. Formally,

$$\Pi^* = \{I_1, I_2 : I_1 = I_1(c, k), I_2 = I_2(c, k), (c, k) \in \Pi\} \tag{2.152}$$

According to the definition of the Π set, the values of I_1 and I_2 belonging to Π^* are related by

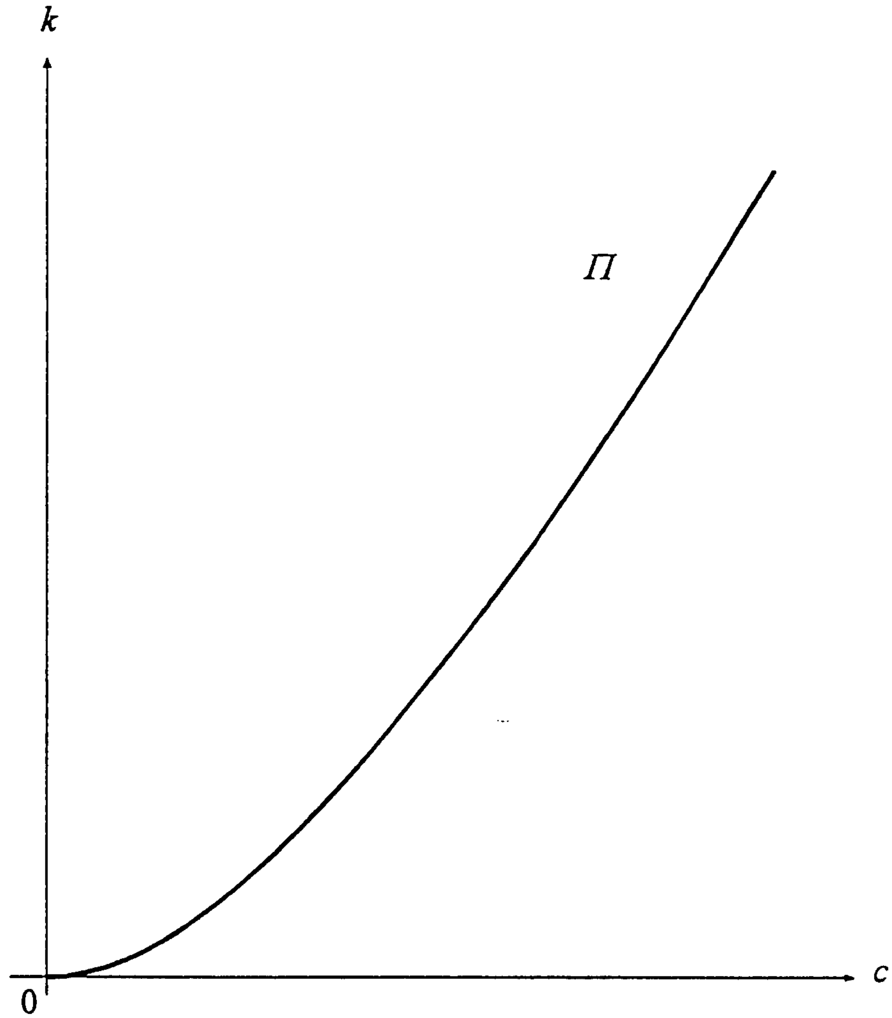


Figure 2-8. Pareto-optimal set in the design variable plane for an isolator with linear spring and quadratic law damper.

Eqs. (2.132) or (2.134). Eliminating D from Eq.(2.132) or U from Eq. (2.134) we arrive at the following relationship between I_1 and I_2:

$$I_2 = \frac{\mu\beta^2}{I_1}, \quad \mu = I_2[(c^0)', (k^0)']. \tag{2.153}$$

Although $(c^0)'$ and $(k^0)'$ are not necessarily uniquely determined (as, for example, at $r = 0$ and $n = 0$), the parameter μ in Eq. (2.153) is determined uniquely since $I_2[(c^0)', (k^0)']$ is the absolute minimum of the peak acceleration in Problem 2.4 in the case of $\beta = 1$ and $D = 1$. Hence, for isolators with power law characteristics, the set Π^* consists of only one curve (hyperbola) of Eq. (2.153), provided n and r are fixed. However, the values of $I_2[(c^0)', (k^0)']$ are different for different exponents n and r of the isolator damping and stiffness characteristics. Therefore, to each pair of n and r there corresponds a special hyperbola of Eq. (2.153). Below all of them lies the hyperbola corresponding to the limiting isolation capabilities, with $I_2[(c^0)', (k^0)'] = 0.5$. Figure 2.9 shows the Π^* sets for isolators with a linear spring and dampers with various exponents r. The curves in this figure correspond to $\beta = 1$. As follows from Eq. (2.153), the value of the initial velocity β does not change qualitatively the relative arrangement of the Π^* sets.

The Π^* set possesses the property that for r and n specified, it is impossible to achieve the values of I_1 and I_2 lying below the corresponding hyperbola of Eq. (2.153). It is not possible to make I_1 and I_2 fall below the hyperbola $I_2 = \beta^2/(2I_1)$, which corresponds to the limiting isolation capabilities.

2.3.7 Isolator with a Dry-Friction Damper.

2.3.7.1 Fundamentals of Dry Friction. For the sake of completeness, we choose to describe dry friction or, more precisely, the Coulomb model of dry friction. For simplicity, we confine our consideration to a single-degree-of-freedom motion of a body along a line. Consider a body which can move on a plane surface (for example, the horizontal surface) along a line l. Introduce a Cartesian coordinate system Oxy so that the x-axis lies on the line l and the y-axis is perpendicular to the x-axis and points to the half-space where the body is located. Define the *active force* as any force applied to the body apart from that due to the interaction of the body with the supporting surface. The latter force will be referred to as the *reaction* or *constraint* force. Let F be the x-component of the active force applied to the body and N the magnitude of the y-component of the active force. Note that the y-component of the active force must be directed toward the supporting surface so as to press the body against this surface. Sometimes, the quantity N is referred to as the normal force. The supporting surface reacts to the active force with the constraint force $\mathbf{R}$. Denote by R_x and R_y the x- and y-components of the constraint force, respectively. The component R_y is always equal to N so that the resultant of the forces normal to the surface is zero. The component R_x which is referred to as the *force of friction* (*friction force*) depends on the velocity of motion of the body, physical parameters characterizing the contact of the supporting surface with the surface of the body, and conditions of loading (i.e., the components of the active force).

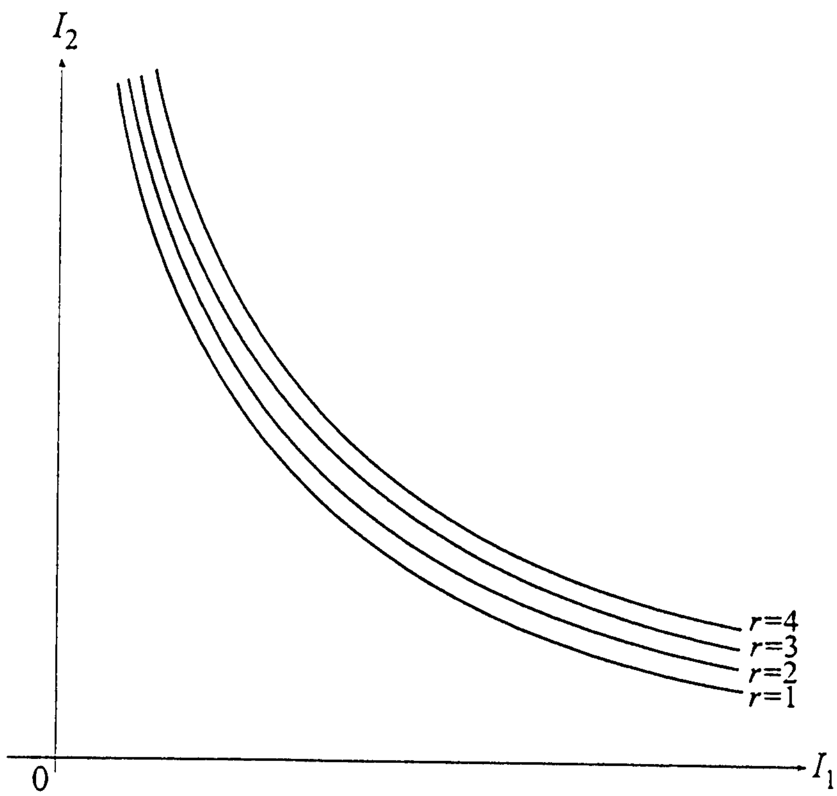


Figure 2-9. Pareto-optimal sets in the space of performance criteria for a linear spring and various dampers.

According to the Coulomb model of dry friction, this dependence has the form

$$R_x = \begin{cases} -\mu N \operatorname{sign}(\dot{x}), & \text{if} \quad \dot{x} \neq 0 \\ -F, & \text{if} \quad \dot{x} = 0 \quad \text{and} \quad |F| \leq \mu N \\ -\mu N \operatorname{sign}(F), & \text{if} \quad \dot{x} = 0 \quad \text{and} \quad |F| > \mu N \end{cases}, \tag{2.154}$$

where x is the coordinate of the body, $\dot{x}$ is the velocity of the body, and μ is a positive factor called the *friction factor* (*coefficient of friction*).

From Eq. (2.154), it is apparent that if the velocity of the body is nonzero ($\dot{x} \neq 0$), then the friction force points to the direction opposite to that of the velocity and its magnitude is equal to μN, irrespective of the velocity magnitude. If the velocity is zero but the x-component F of the active force does not exceed μN ($\dot{x} = 0$ and $|F| \leq \mu N$), the friction force is equal in magnitude but opposite in sign to F. Hence, the resultant force acting on the body in the x-direction is zero, and if the body rests, one cannot make it move unless the magnitude of F exceeds the value μN. Sometimes, the quantity μN is referred as the maximum static friction force. Finally, if the velocity of the body is zero and the magnitude of the force F exceeds the maximum static friction force ($\dot{x} = 0$ and $|F| > \mu N$), then the friction force acts in the direction opposite to that of F with the magnitude of the maximum static friction force.

Note that in the Coulomb friction model of Eq. (2.154), the magnitude of the kinetic friction force (for $\dot{x} \neq 0$) coincides with the maximum static friction force. In reality, the situation is more complicated. The magnitude of the kinetic force depends on the velocity, for small velocities the magnitude of the kinetic friction force being less than the maximum static friction force. For more detail, see, for example, Shelley (1980). However, this dependence is relatively weak and can be neglected in many practical analyses. In what follows, we consider only the Coulomb model of dry friction and use the terms dry friction, Coulomb friction, and Coulomb damping as synonyms.

2.3.7.2 Example 2.4. *Equation of motion of an oscillator with a Coulomb damper.* Consider a body which is attached to a fixed base by a linear spring and can translate along the x-direction of the rough horizontal plane. Let m be the mass of the body, x the displacement of the body with respect to the position corresponding to the undeformed spring, K the spring coefficient of stiffness, μ the coefficient of friction between the contact surfaces of the body and the supporting plane, and g the acceleration due to gravity. For this system, the active forces are the force $F = -Kx$ exerted on the body in the horizontal direction by the spring and the gravitational force $N = mg$ acting vertically downward. These forces are identified with the forces F and N, respectively, in the expression of Eq. (2.154) for dry friction force. Thus,

$$R_x = \begin{cases} -\mu mg \operatorname{sign}(\dot{x}), & \text{if} \quad \dot{x} \neq 0 \\ Kx, & \text{if} \quad \dot{x} = 0 \quad \text{and} \quad |Kx| \leq \mu mg \\ \mu mg \operatorname{sign}(x), & \text{if} \quad \dot{x} = 0 \quad \text{and} \quad |Kx| > \mu mg \end{cases} \tag{2.155}$$

According to Newton's second law, the equation of motion the body along the x-axis is

$$m\ddot{x} = F + R_x \tag{2.156}$$

or, since $F = -Kx$,

$$m\ddot{x} = -Kx + R_x. \tag{2.157}$$

where R_x has the form of (2.155). Denote $k = K/m$ and $c = \mu g$ and divide both sides of (2.157) by m. This leads to

$$\ddot{x} + kx + cq = 0, \tag{2.158}$$

where

$$q = \begin{cases} \operatorname{sign}(\dot{x}), & \text{if } \dot{x} \neq 0 \\ -kx/c, & \text{if } \dot{x} = 0 \text{ and } |kx| \leq c \\ -\operatorname{sign}(x), & \text{if } \dot{x} = 0 \text{ and } |kx| > c \end{cases}. \tag{2.159}$$

Equation (2.158) describes oscillations of a body on a spring damped by a Coulomb damper. For example, this equation can describe the transient motion of an isolated object excited by an impulsive impact if the shock isolator consists of a spring with a linear characteristic and a Coulomb damper. See Eq. (2.161) in the following subsection. It is appropriate to refer to the coefficient c in (2.158) as the (Coulomb) damping coefficient.

2.3.7.3 Specific Features of the Behavior of Systems with a Dry-Friction Damper and Formulation of the Optimization Problem. Consider in more detail the isolator composed of a linear spring and a dry-friction (Coulomb) damper. The characteristic $u(x, \dot{x})$ of such an isolator is given by

$$u(x, \dot{x}) = cq + kx, \tag{2.160}$$

$$q = \begin{cases} \operatorname{sign}(\dot{x}), & \text{if } \dot{x} \neq 0 \\ -kx/c, & \text{if } \dot{x} = 0 \text{ and } |kx| \leq c \\ -\operatorname{sign}(x), & \text{if } \dot{x} = 0 \text{ and } |kx| > c \end{cases}.$$

Here c is the Coulomb damping factor and k is the stiffness coefficient of the spring. The characteristic of Eq. (2.160) coincides with the one given by Eq. (2.112) with $r = 0$ and $n = 0$, except for $\dot{x} = 0$. The specific feature of this isolator characteristic is that, unlike systems governed by the equation

$$\ddot{x} + c|\dot{x}|^r \operatorname{sign}(\dot{x}) + k|x|^n \operatorname{sign}(x) = 0, \quad r > 0, n > 0,$$

the system

$$\ddot{x} + cq + kx = 0 \tag{2.161}$$

corresponding to the characteristic of Eq. (2.160) has not only the equilibrium position $x = 0$ but a continuum of equilibrium positions, the so-called *stick zone* given by

$$C = \{x : |kx| \leq c\}. \tag{2.162}$$

If at some instant $t = t^*$ the conditions $|kx(t^*)| \leq c$ and $\dot{x}(t^*) = 0$ are fulfilled, then, as follows from Eqs. (2.161) and (2.160), $x(t) = x(t^*)$ for all $t \geq t^*$. The presence of a stick zone is characteristic of oscillatory systems subject to dry friction.

As has been shown in Section 2.3.4, the parameters

$$c = c^0 = \frac{\beta^2}{2D}, \qquad k = k^0 = 0 \tag{2.163}$$

are optimal parameters solving Problem 2.4 for the system of Eq. (2.161), while

$$c = c_0 = U, \qquad k = k_0 = 0 \tag{2.164}$$

are optimal parameters solving Problem 2.3. In these cases, the limiting isolation capabilities evaluated by Eqs. (2.28) or (2.29) are achieved. Note that in Problems 2.3 and 2.4, it was assumed that the body being isolated was at the position $x = 0$ at the instant of impact (see the initial conditions in Eq. (2.32)). This assumption is justified for the systems for which the position $x = 0$ is a single equilibrium position and, moreover, this equilibrium is asymptotically stable. However, the motion of the system of Eq. (2.161) under the initial conditions $x(0) = 0, \dot{x}(0) = \beta$ and $k = 0$ is given by

$$x(t) = \begin{cases} \beta t - \frac{ct^2}{2}\operatorname{sign}(\beta), & \text{if} \quad 0 \le t \le |\beta|/c \\ \frac{\beta|\beta|}{2c}, & \text{if} \quad t > |\beta|/c \end{cases} \tag{2.165}$$

and the body being isolated having attained its extremum displacement $x = \beta\,|\beta|\,/(2c)$ will remain in this position and will not return to the initial position $x = 0$.

In the general case, shock isolators with power law characteristics given by Eq. (2.112) for $r > 0$ provide the asymptotic return of the body being isolated to the equilibrium position $x = 0$ after the action of an external disturbance has ceased. An isolator with a dry-friction damper does not possess such a property and if the system is subjected to shock disturbances repeatedly, the body being isolated can find itself at any position within the stick zone of Eq. (2.162) at the instant of shock. Therefore, when calculating optimal parameters of an isolator with the characteristic of Eq. (2.160) it is reasonable to assume that just before the shock the body being isolated rests at the least favorable position within the stick zone C.

Denote by $x(t;\xi,c,k)$ the solution of Eq. (2.161) under initial conditions $x(0) = \xi$, $\dot{x}(0) = \beta$. As performance criteria we take the maximum value of the peak displacement of the body being isolated,

$$I_1(c,k) = \max_{\xi \in C} \max_{t \in [0,\infty)} |x(t;\xi,c,k)| \tag{2.166}$$

and the maximum value of its peak acceleration,

$$I_2(c,k) = \max_{\xi \in C} \max_{t \in [0,\infty)} |\ddot{x}(t;\xi,c,k)|. \tag{2.167}$$

Consider the problem of determining optimal parameters $c = c^0$ and $k = k^0$ minimizing the criterion of Eq. (2.167) under the constraint on the criterion of Eq. (2.166):

$$I_2(c^0, k^0) = \min_{c,k} I_2(c, k), \quad I_1(c, k) \le D, \quad c \ge 0, k \ge 0. \tag{2.168}$$

The problem of Eq. (2.168) is a generalization of Problem 2.4 for the case where the initial position of the body is not specified beforehand but is assumed to belong to the uncertainty set C. It corresponds to Problem 2.2, where Y is the two-parameter family of functions of Eq. (2.160), while $G_1 = \{x, \dot{x} : |x| \le c/k, \dot{x} = 0\}$.

2.3.7.4 Solution of the Optimization Problem. A simple analysis of the motion governed by Eq. (2.161) subject to the initial conditions $x(0) = \xi$ and $\dot{x}(0) = \beta$ shows that if $\xi \in C$, then the peak acceleration, $\max_t |\ddot{x}(t; \xi, c, k)|$, and the peak displacement, $\max_t |x(t; \xi, c, k)|$, occur simultaneously, at the instant $t = t_*$, when the velocity $\dot{x}(t; \xi, c, k)$ vanishes for the first time after the shock. On the interval $[0, t_*]$, the motion of the body being isolated is described by the linear differential equation with initial conditions

$$\ddot{x} + kx = -c\,\mathrm{sign}(\beta), \quad x(0) = \xi, \qquad \dot{x}(0) = \beta, \quad t \in [0, t_*]. \tag{2.169}$$

Solving this initial-value problem and calculating the functionals of Eqs. (2.166) and (2.167) we obtain

$$I_1(c, k) = \left(\frac{4c^2}{k^2} + \frac{\beta^2}{k}\right)^{1/2} - \frac{c}{k}, \tag{2.170}$$

$$I_2(c, k) = (4c^2 + k\beta^2)^{1/2}. \tag{2.171}$$

Direct differentiation shows that derivatives of $I_2(c, k)$ with respect to c and k are positive. Hence, optimal values of c and k lie on the boundary of the admissible region for the parameters. According to Eqs. (2.168) and (2.170), one of the portions of the boundary is the curve given by

$$I_1(c, k) = \left(\frac{4c^2}{k^2} + \frac{\beta^2}{k}\right)^{1/2} - \frac{c}{k} = D. \tag{2.172}$$

Solving this equation for k leads to a single positive root

$$k = k(c) = \frac{\beta^2 - 2cD + [(\beta^2 - 2cD)^2 + 12c^2D^2]^{1/2}}{2D^2}. \tag{2.173}$$

We denote by γ_1 the curve specified by Eq. (2.173) . The analysis of the function of Eq. (2.173) for $0 \le c < \infty$ shows that this function (1) is equal to β^2/D^2 at $c = 0$, (2) monotonically decreases on the interval $[0, \beta^2/(4D))$, (3) attains the minimum value equal to $3\beta^2/(4D^2)$ at $c = \beta^2/4D$, and (4) monotonically grows on the interval $(\beta^2/(4D), \infty)$, $k(c) \sim [\beta^2 + 2cD]/(2D^2)$ as $c \to \infty$. Points of the ck-plane lying either above the curve γ_1 or on this curve (and only these points) satisfy the inequality $I_1(c, k) \le D$ (Eq. (2.168)). The aforementioned properties of the curve γ_1 imply that

the boundary of the admissible region of the parameters c and k in the problem of Eq. (2.168) consists of the curve γ_1 and a rectilinear portion, the ray $\{c, k : c = 0, \quad \beta^2/D^2 < k < \infty\}$. Since the function $I_2(c,k)$ monotonically increases with respect to k, the optimal parameters c^0 and k^0 desired in the problem of Eq. (2.168) cannot belong to the aforementioned ray and hence, lie on the curve γ_1. Analysis shows that the function $I_2(c, k(c))$, where $k(c)$ is specified by Eq. (2.173), has a unique minimum over the set $0 \le c < \infty$. This minimum occurs at $c = c^0 = \beta^2/(8D)$ and is equal to $\beta^2\sqrt{(7+2\sqrt{11})}/(4D)$. The corresponding value of the parameter k is equal to $k(c^0) = \beta^2(3+\sqrt{11})/(8D^2)$.

Thus, the optimal parameters in the problem of Eq. (2.168) are given by

$$c^0 = \frac{\beta^2}{8D}, \quad k^0 = \frac{\beta^2}{8D^2}(3+\sqrt{11}). \tag{2.174}$$

The corresponding values of the performance criterion to be minimized, $I_2(c,k)$, and the constrained performance criterion, $I_1(c,k)$, are

$$I_2(c^0,k^0) = \frac{\beta^2}{4D^2}\sqrt{(7+2\sqrt{11})} \approx 0.923\frac{\beta^2}{D}, \quad I_1(c^0,k^0) = D. \tag{2.175}$$

As is seen from Eq. (2.175), under the optimal parameters given by Eq. (2.174), the criterion to be minimized, I_2, monotonically decreases with the growth of the maximum allowable displacement D, while the peak displacement, $I_2(c^0,k^0)$ is exactly D. Therefore, according to Theorem 1.1, the problem of Eq. (2.168) is the reciprocal of the problem of determining the optimal parameters $c = c_0$ and $k = k_0$ minimizing the maximum displacement $I_1(c,k)$, provided the constraint $I_2(c,k) \le U$ is imposed on the peak acceleration of the body. The solution to the latter problem is given by

$$c_0 = \frac{U}{2\sqrt{(7+2\sqrt{11})}} \approx 0.135U, \quad k_0 = \frac{2U^2}{\beta^2}\frac{3+\sqrt{11}}{7+2\sqrt{11}} \approx 0.927\frac{U^2}{\beta^2}; \tag{2.176}$$

$$I_1(c_0,k_0) = \frac{\beta^2}{4U}\sqrt{(7+2\sqrt{11})} \approx 0.923\frac{\beta^2}{U}, \quad I_2(c_0,k_0) = U. \tag{2.177}$$

2.4 CONCLUSIONS AND PRACTICAL RECOMMENDATIONS.

In this chapter we have established the limiting capabilities of shock isolation for rectilinearly moving single-degree-of-freedom systems. They are characterized by

$$J_2(u^0) = \frac{\beta^2}{2D}, \qquad J_1(u^0) = D, \tag{2.178}$$

for the case in which the peak acceleration (dynamic load) of the body being isolated is minimized, provided that the maximum displacement is less that a constant D, or

$$J_1(u_0) = \frac{\beta^2}{2U}, \qquad J_2(u_0) = U, \tag{2.179}$$

if the maximum displacement is minimized while the peak acceleration is constrained by a constant U. See Eqs. (2.28) and (2.29). When deriving the formulas of Eqs. (2.178) and (2.179), we assumed that the base is subject to a single instantaneous (impulse) shock of intensity β.

Some types of passive isolators with power law characteristics implement the limiting shock isolation capabilities. These include

(1) The isolator with a linear spring and a quadratic damper for which the damping and stiffness factors are given by

$$\begin{array}{ll} c^0 = \frac{1}{2D}, & k^0 = \frac{\beta^2}{2D^2} \\ (c_0 = \frac{U}{\beta^2}, & k_0 = \frac{2U^2}{\beta^2}); \end{array} \tag{2.180}$$

(2) Undamped isolator with a bang-bang spring of the stiffness

$$k^0 = \frac{\beta^2}{2D} \qquad (k_0 = U); \tag{2.181}$$

(3) The isolator consisting of a dry-friction damper and no stiffness element, the friction factor being

$$c^0 = \frac{\beta^2}{2D} \qquad (c_0 = U); \tag{2.182}$$

(4) The isolator with a bang-bang spring and a dry-friction damper, for which the stiffness and friction factors are related by

$$c^0 + k^0 = \frac{\beta^2}{2D} \quad (c_0 + k_0 = U). \tag{2.183}$$

The formulas of Eqs. (2.180) to (2.183) not enclosed in parentheses correspond to the minimization of the peak acceleration of the body, while the formulas in parentheses are related to the minimization of the maximum displacement.

No isolators with power law characteristics, other than those of (1) to (4), can provide the limiting capabilities of shock isolation.

The isolators of (1) to (4) ensure the same shock isolation quality characterized by Eqs. (2.178) or (2.179), provided at the instant of shock the body being isolated is at rest relative to the base at

the position $x = 0$. From this point of view, all the isolators of (1) to (4) are equivalent. However, in practice, a system may undergo several shocks separated by a significant time interval rather than a single shock. In this case, to guarantee the protection quality characterized by Eqs. (2.178) or (2.179) it is essential that at the instant of each shock the body being isolated is at rest with respect to the base at the position $x = 0$, as is the case at the instant of the initial shock. Of the isolators of (1) to (4), the isolator with a linear spring and quadratic damper and the isolator with a bang-bang spring and a dry friction damper with $c < k$ provide the asymptotic return of the body being isolated to the equilibrium position $x = 0$. For these isolators the condition formulated above is satisfied if the time lag between shocks is sufficiently large. The other isolators do not possess such a property. The use of the undamped isolator with a bang-bang spring gives rise to undamped oscillations of the body being isolated. When using the isolator containing a bang-bang spring and a dry friction damper with $c \geq k$, the body will stop at the position where its velocity vanishes for the first time after the shock. Thus, for systems that may be subject to repeated shock loading, the isolators of (2), (3), and (4) with $c \geq k$ do not guarantee the limiting isolation quality represented by Eqs. (2.178) and (2.179). For this reason we think that it is not expedient to use such isolators without special devices ensuring the return of the body being isolated to its initial position after each shock. However, the introduction of such devices will make the system more expensive in design and operation. For the isolators with a bang-bang spring and a dry friction damper for $c < k$, implementation by simple mechanical elements without including additional (e.g. electromagnetic) control units is complicated because of the bang-bang spring.

Thus, it can be concluded that for a system subject to an impulse disturbance, the isolator with a linear spring and a quadratic damper should be preferable to other types of isolators. Such an isolator is in widespread use in practice and is easy to implement with simple mechanical elements.

We have shown that very good shock protection quality corresponding to

$$I_2(c^0, k^0) = 0.521\frac{\beta^2}{D} \quad (I_1(c_0, k_0) = 0.521\frac{\beta^2}{U}) \tag{2.184}$$

is ensured by a widely used linear isolator, provided its parameters are optimally chosen. According to the optimization criteria, the isolation quality given by this isolator is only 4% lower than the limiting capabilities. Just as for the isolator with a linear spring and a quadratic damper, the linear isolator provides the asymptotic return of the body being isolated to its initial position after a shock and can be successfully used in practice for protecting various devices from shock disturbances.

In Section 2.3.6, the isolator with a linear spring and a dry-friction damper has been investigated for shock protection potential. The presence of a stick zone in this system significantly increases the optimal value of the performance index to be minimized. This value, given by Eqs. (2.175) or (2.177), is 84.6% greater than that for the limiting isolation capabilities and 80.4% greater than the optimal value corresponding to the linear isolator. Therefore, the use of the isolator with a dry friction damper and a linear spring for shock protection does not appear to be advisable.

SECTION 3

OPTIMAL PROTECTION OF ROTATING OBJECTS FROM AN INSTANTANIOUS IMPACT

3.1 PROBLEM FORMULATION.

Consider a mechanical system which is represented by a rigid body (the object to be isolated) rotating about a fixed axis (Fig. 3.1). The body is connected to the base by means of an isolator generating the control torque with respect to the axis of rotation. We assume that at an initial instant $t = 0$, the body acquires an angular velocity β resulting from a shock. The motion of the system is governed by the differential equation with initial conditions

$$I\ddot{\varphi} = M(\varphi, \dot{\varphi}, t), \quad \varphi(0) = 0, \quad \dot{\varphi}(0) = \beta. \tag{3.1}$$

Here, I is the moment of inertia of the body with respect to the axis of rotation, φ is the angle of rotation, and $M(\varphi, \dot{\varphi}, t)$ is the control torque (the isolator characteristic).

Typical performance criteria for the shock isolation of rotating objects are the maximum (over time) absolute value of the acceleration of some point of the rotating body (this criterion characterizes the load transmitted to the body at this point) and the maximum angle of rotation.

The square of the absolute value of the acceleration of the rigid body point located at a distance of ρ from the axis of rotation is given by

$$a^2(\rho, t) = \rho^2(\ddot{\varphi}^2 + \dot{\varphi}^4) = \rho^2(M^2/I^2 + \dot{\varphi}^4). \tag{3.2}$$

To derive Eq. (3.2), recall that the acceleration of a point P of a rigid body rotating about a fixed axis can be developed into two components, which are the *tangential acceleration* $\mathbf{a}_t$ and the *centripetal acceleration* $\mathbf{a}_c$ (Fig. 3.2). The tangential and the centripetal accelerations lie in the plane Π which passes through the point P and is perpendicular to the axis of rotation. The tangential acceleration is perpendicular to the position vector $\boldsymbol{\rho}$ of the point P in the plane Π and its magnitude is

$$a_t = |\mathbf{a}_t| = \rho|\ddot{\varphi}|, \tag{3.3}$$

where $\rho = |\boldsymbol{\rho}|$ is the distance from the axis of rotation to the point P and $\ddot{\varphi}$ is the angular acceleration of the rotation of the body about the axis. The centripetal acceleration is directed along the position vector $\boldsymbol{\rho}$ toward the axis of rotation. The magnitude of the centripetal acceleration is

$$a_c = |\mathbf{a}_c| = \rho\dot{\varphi}^2, \tag{3.4}$$

where $\dot{\varphi}$ is the angular velocity of the body. The total acceleration $\mathbf{a}$ of the point is the vector sum

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_c \tag{3.5}$$

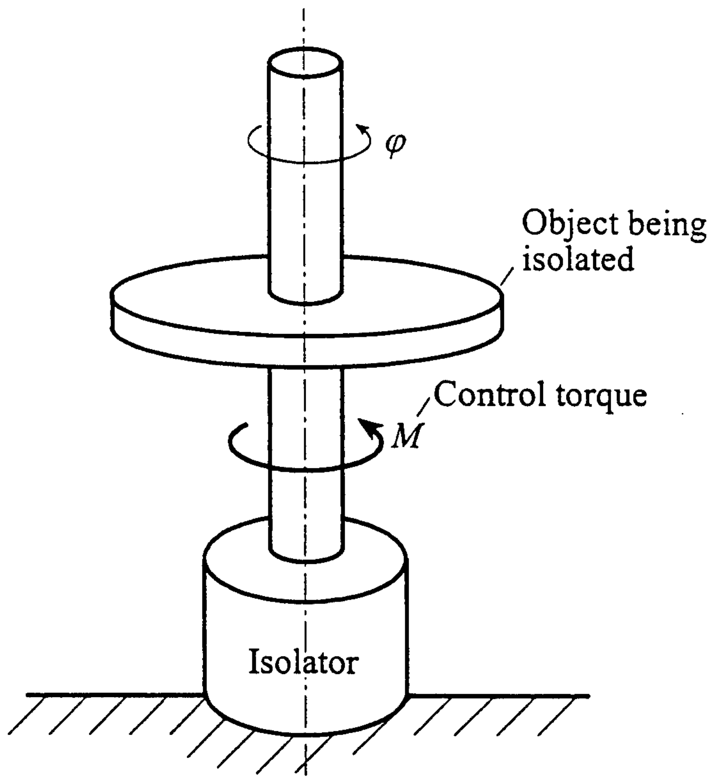


Figure 3-1. Rotating object connected to the base by means of an isolator.

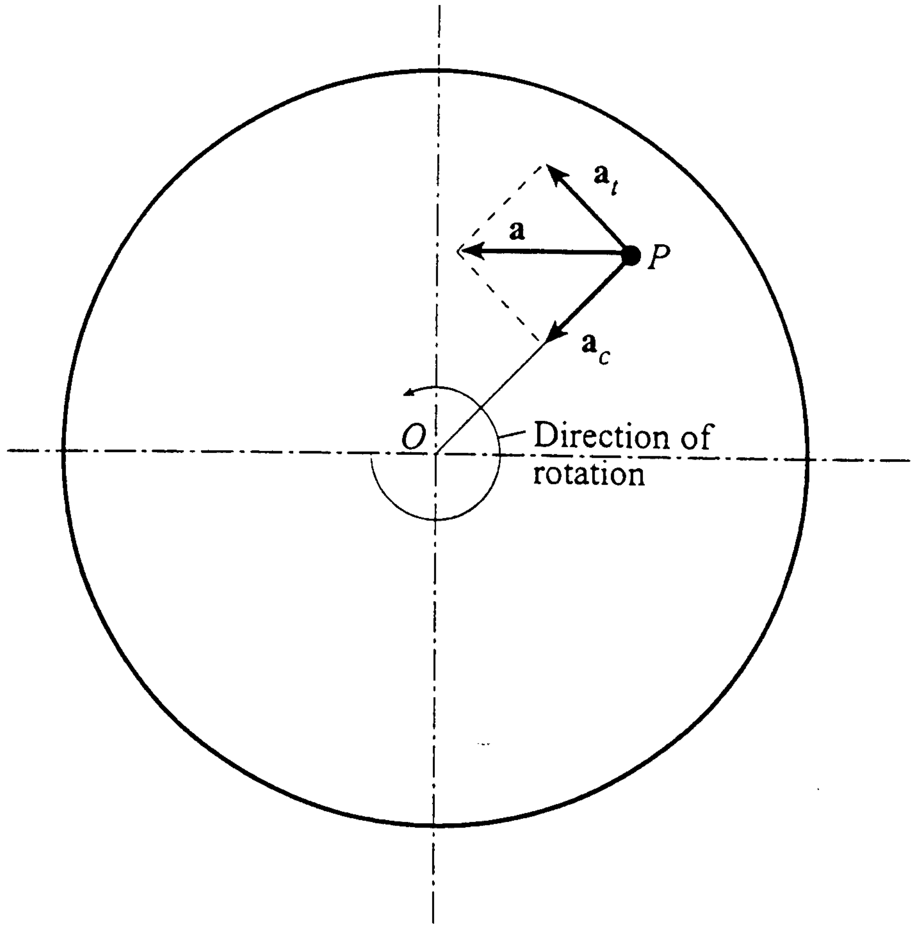


Figure 3-2. Tangential and centripetal acceleration of a point on a rotating body. $\mathbf{a}_t$: tangential acceleration; $\mathbf{a}_c$: centripetal acceleration; $\mathbf{a}$: total acceleration.

of the tangential and the centripetal components. Since the terms $\mathbf{a}_t$ and $\mathbf{a}_c$ are perpendicular to each other, the absolute value a of the resultant vector $\mathbf{a}$ is

$$a = |\mathbf{a}| = |\mathbf{a}_t + \mathbf{a}_c| = \sqrt{a_t^2 + a_c^2}. \tag{3.6}$$

Substitution of Eqs. (3.3) and (3.4) into Eq. (3.6) yields

$$a = \rho\sqrt{\ddot{\varphi}^2 + \dot{\varphi}^4}. \tag{3.7}$$

or

$$a^2 = \rho^2(\ddot{\varphi}^2 + \dot{\varphi}^4). \tag{3.8}$$

From Eq. (3.1), express $\ddot{\varphi}$ in terms of M and I and substitute the resulting relation into Eq. (3.8). This yields

$$a^2 = \rho^2(M^2/I^2 + \dot{\varphi}^4). \tag{3.9}$$

Equations (3.8) and (3.9) lead to Eq. (3.2).

$$t' = |\beta|\, t, \quad \varphi' = \varphi\, \text{sign}(\beta), \quad u = -M/(I\beta^2). \tag{3.10}$$

Then Eqs. (3.1) and (3.2) can be represented as

$$\ddot{\varphi} + u(\varphi, \dot{\varphi}, t) = 0, \quad \varphi(0) = 0, \quad \dot{\varphi}(0) = 1 \tag{3.11}$$

$$a^2(\rho, t) = \rho^2\beta^4\left[u^2 + \dot{\varphi}^4(t; u)\right]. \tag{3.12}$$

Primes indicating the dimensionless variables are omitted. Henceforth, we will denote by $\varphi(t; u)$ the solution of the initial-value problem of Eq. (3.11) corresponding to the control u.

We introduce the functionals

$$J_1(u) = \max_{t\in[0,\infty)} |\varphi(t; u)| \tag{3.13}$$

$$J_2(u) = \max_{t\in[0,\infty)} \left\{u^2[\varphi(t; u), \dot{\varphi}(t; u), t] + \dot{\varphi}^4(t; u)\right\} \tag{3.14}$$

and formulate two optimization problems.

3.1.1.1 Problem 3.1. For the system governed by the initial-value problem of Eq. (3.11), find a control (isolator characteristic) $u_U^0(\varphi, \dot{\varphi}, t) \in Y$ such that

$$J_1(u_U^0) = \min_{u\in Y} J_1(u), \quad J_2(u) \leq U. \tag{3.15}$$

3.1.1.2 Problem 3.2. For the system governed by the initial-value problem of Eq. (3.11), find a control $u_0^D(\varphi, \dot{\varphi}, t) \in Y$ such that

$$J_2(u_0^D) = \min_{u \in Y} J_2(u), \quad J_1(u) \leq D, \tag{3.16}$$

where Y is a specified set of admissible isolator characteristics, and U and D are prescribed positive numbers.

Problem 3.1 corresponds to minimizing the maximum absolute value of the angle of rotation of the body subject to a constraint on the maximum load at a given point. On the other hand, Problem 3.2 reflects the desire to minimize the maximum load at an arbitrary point of the object being isolated, provided the maximum absolute value of the angle of rotation is constrained. As follows from Eqs. (3.12) and (3.14), $U = W/(\rho^2\beta^4)$ where W is the square of the maximum allowable acceleration at the point lying at a distance of ρ from the axis of rotation.

Note that Problem 3.1 has no solution for $U < 1$, since at $t = 0$ we have $u^2 + \dot{\varphi}^4(0; u) = u^2 + 1 \geq 1$ and, hence, the inequality $J_2 \leq U$ cannot be satisfied for any u.

3.2 LIMITING ISOLATION CAPABILITIES.

Consider the limiting isolation capability problem conforming to Problem 3.1. Assume Y is a set of piecewise-continuous functions of time t, continuous on the right at discontinuity points.

Since, according to Eq. (3.11), the angular velocity $\dot{\varphi}$ is positive at the initial time instant, the function $\varphi(t; u)$ for any admissible control $u(t)$ monotonically increases in the interval $[0, T)$, where T is the first instant at which the angular velocity vanishes. As follows from Eq. (3.11), for controls $u(t)$ identically equal to zero outside the segment $[0, T]$, the maximum absolute value of the angle of rotation is equal to $\varphi(T; u)$, because in this case $\varphi(t; u) \equiv \varphi(T; u)$ for $t \geq T$. Since the inequality $J_1(u) \geq \varphi(T; u)$ is valid for any $u(t)$, the limiting capabilities problem can be formulated as follows: *Find an admissible control* $u_1^U(t)$ *defined over the segment* $[0, T]$ *such that*

$$\varphi(T; u_1^U) = \min_u \varphi(T; u), \quad \dot{\varphi}(T; u) = 0, \tag{3.17}$$

$$u^2(t) + \dot{\varphi}^4(t; u) \leq U. \tag{3.18}$$

Note that the terminal instant T is not fixed beforehand; it is to be determined when solving the problem.

If $u_1^U(t)$ is the optimal control in the problem of Eqs. (3.17) and (3.18), then the control

$$u_U^0 = \begin{cases} u_1^U, & t \in [0, T) \\ 0, & t \notin [0, T] \end{cases} \tag{3.19}$$

is optimal for Problem 3.1 considered for the infinite time interval $[0, \infty)$.

3.2.1.1 Proposition 3.1. Let $u_1^U(t)$ be the optimal control and T be the corresponding instant at which the angular velocity vanishes for the first time. Then the equality $\left[u_1^U(t)\right]^2 + \dot{\varphi}^4(t; u_1^U) = U$ is satisfied identically over the segment $[0, T]$.

3.2.1.2 Proof. Denote by A_T the set of admissible controls that satisfy the inequality of Eq. (3.18) and for which the angular velocity vanishes at the instant T. The solution of the initial-value problem of Eq. (3.11) in the case where u is a function of time only can be represented as

$$\varphi(t; u) = t - \int_0^t (t - \tau) u(\tau) d\tau, \tag{3.20}$$

$$\dot{\varphi}(t; u) = 1 - \int_0^t u(\tau) d\tau. \tag{3.21}$$

Suppose Proposition 3.1 is not true, i.e. there exists an instant $t^* \in [0, T]$ such that $\left[u_1^U(t^*)\right]^2 + \dot{\varphi}^4(t^*; u_1^U) < U$. Then, since the control is a piecewise-continuous function, continuous on the right at discontinuity points, there exists an interval $[t_1, t_2) \subset [0, T]$ such that

$$\left[u_1^U(t)\right]^2 + \dot{\varphi}^4(t; u_1^U) < U - \delta, \quad t \in [t_1, t_2) \tag{3.22}$$

where δ is a sufficiently small positive number.

We construct the control $\tilde{u}(t)$ as follows:

$$\tilde{u}(t) = \begin{cases} u_1^U(t), & \text{for} \quad t \notin [t_1, t_2); \\ u_1^U(t) + \varepsilon, & \text{for} \quad t \in [t_1, \frac{t_1+t_2}{2}); \\ u_1^U(t) - \varepsilon, & \text{for} \quad t \in [\frac{t_1+t_2}{2}, t_2) \end{cases} \tag{3.23}$$

where ε is a positive number. The control of Eq. (3.23) is admissible (i.e. piecewise-continuous, continuous on the right at discontinuity points) if $u_1^U(t)$ is an admissible control. Equations (3.20), (3.21) and (3.23) imply that

$$\dot{\varphi}(t; u_1^U) = \dot{\varphi}(t; \tilde{u}), \quad \text{for} \quad t \notin [t_1, t_2); \tag{3.24}$$

$$\varphi(T; \tilde{u}) = \varphi(T; u_1^U) - \varepsilon \int_{t_1}^{(t_1+t_2)/2} (T - \tau) d\tau + \varepsilon \int_{(t_1+t_2)/2}^{t_2} (T - \tau) d\tau =$$

$$= \varphi(T; u_1^U) - \frac{\varepsilon}{4}(t_2 - t_1)^2 < \varphi(T; u_1^U). \tag{3.25}$$

Let us show that one can always choose the parameter ε in Eq. (3.23) in such a way that $\tilde{u}(t) \in A_T$. Since $\dot{\varphi}(t; u_1^U) = \dot{\varphi}(t; \tilde{u})$ for $t \notin [t_1, t_2)$, the relations $\dot{\varphi}(T; u_1^U) = \dot{\varphi}(T; \tilde{u}) = 0$ hold and the inequality of Eq. (3.18) for $u = \tilde{u}$ is satisfied for $t \notin [t_1, t_2)$.

It remains to prove that the inequality of Eq. (3.18) is satisfied also in the interval $[t_1, t_2)$ for some ε. For $t \in [t_1, \frac{t_1+t_2}{2})$ we have

$$\tilde{u}^2(t) + \dot{\varphi}^4(t;\tilde{u}) = \left[u_1^U(t) + \varepsilon\right]^2 + \left[\dot{\varphi}(t;u_1^U) + \varepsilon(t-t_1)\right]^4 =$$

$$= \left[u_1^U(t)\right]^2 + \dot{\varphi}^4(t;u_1^U) + \sum_{i=1}^{4} \varepsilon^i \Psi_i(t), \tag{3.26}$$

where

$$\Psi_1(t) = 2u_1^U(t) - 4\dot{\varphi}^3(t;u_1^U)(t-t_1),$$

$$\Psi_2(t) = 1 + 6\dot{\varphi}^2(t;u_1^U)(t-t_1)^2,$$

$$\Psi_3(t) = 4\dot{\varphi}(t;u_1^U)(t-t_1)^3,$$

$$\Psi_4(t) = (t-t_1)^4.$$

The solution $\varphi(t;u_1^U)$ of the initial-value problem of Eq. (3.11) is continuous, as is its time derivative. Hence, functions $\Psi_i(t), i = 1,2,3,4$, have finite upper bounds in the interval $[t_1, \frac{t_1+t_2}{2})$. In this case, Eqs. (3.22) and (3.26) imply that

$$\tilde{u}^2(t) + \dot{\varphi}^4(t;\tilde{u}) < U - \delta + O_1(\varepsilon), \tag{3.27}$$

where

$$O_1(\varepsilon) = \sum_{i=1}^{4} \varepsilon^i \sup_t \Psi_i(t) \to 0 \quad \text{as } \varepsilon \to 0.$$

In a similar manner, considering the interval $[\frac{t_1+t_2}{2}, t_2)$, it can be shown that

$$\tilde{u}^2(t) + \dot{\varphi}^4(t;\tilde{u}) < U - \delta + O_2(\varepsilon). \tag{3.28}$$

$$O_2(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

According to Eqs. (3.27) and (3.28), by choosing ε so that $\max\{|O_1(\varepsilon)|, |O_2(\varepsilon)|\} < \delta$ we ensure the fulfillment of the inequality of Eq.(3.18) for the control $\tilde{u}$ on the interval $[t_1, t_2)$ and hence, on the entire interval $[0, T]$, i.e. $\tilde{u}(t) \in A_T$.

Thus, we have shown that if Proposition 3.1 is not true, then there exists a control $\tilde{u}(t) \in A_T$ satisfying inequality (3.25) and, hence, the control $u_1^U(t)$ is not optimal. This contradiction proves Proposition 3.1.

Proposition 3.1 implies the following relation for the optimal control:

$$u_1^U(t) = \sqrt{U - \dot{\varphi}^4(t; u_1^U)}. \tag{3.29}$$

Substitute Eq. (3.29) for u in Eq. (3.11). This leads to the initial-value problem

$$\ddot{\varphi}(t; u_1^U) = -\sqrt{U - \dot{\varphi}^4(t; u_1^U)} \tag{3.30}$$

$$\varphi(0; u_1^U) = 0, \quad \dot{\varphi}(0; u_1^U) = 1,$$

which governs the optimal motion.

The solution to the problem of Eq. (3.30) in the segment $[0, T]$ is given by

$$\varphi(t; u_1^U) = \tan^{-1}\left[\mathrm{cn}(\sqrt{2\sigma}(T-t), 1/\sqrt{2})\right] - \tan^{-1}\left[\sqrt{(\sigma-1)/(\sigma+1)}\right], \tag{3.31}$$

$$\dot{\varphi}(t; u_1^U) = \frac{\sqrt{\sigma}\,\mathrm{sn}(\sqrt{2\sigma}(T-t), 1/\sqrt{2})}{\sqrt{2 - \mathrm{sn}^2(\sqrt{2\sigma}(T-t), 1/\sqrt{2})}}, \tag{3.32}$$

$$T = \frac{1}{\sqrt{2\sigma}} F\left(\sin^{-1}\sqrt{\frac{2}{\sigma+1}}, \frac{1}{\sqrt{2}}\right), \quad \sigma = \sqrt{U}, \tag{3.33}$$

where $F(\alpha, \kappa)$ denotes the elliptic integral of the first kind, while $\mathrm{sn}(\psi, \kappa)$ and $\mathrm{cn}(\psi, \kappa)$ stand for the elliptic sine and cosine, respectively.

By substituting Eq. (3.32) into Eq. (3.29) we obtain the expression

$$u_1^U(t) = 2\sigma \frac{\mathrm{cn}(\sqrt{2\sigma}(T-t), 1/\sqrt{2})}{2 - \mathrm{sn}^2(\sqrt{2\sigma}(T-t), 1/\sqrt{2})} \tag{3.34}$$

for the optimal control. According to Eq. (3.31), the optimal value of the angle of rotation $\varphi(T; u_1^U)$ is given by

$$\varphi(T; u_1^U) = \frac{\pi}{4} - \tan^{-1}\sqrt{\frac{\sigma - 1}{\sigma + 1}}, \quad \sigma = \sqrt{U}. \tag{3.35}$$

As mentioned previously, the control of Eq. (3.19) is the solution of Problem 3.1 and thus provides the limiting isolation capabilities. Hence, the minimum value of the functional $J_1(u)$ (the maximum absolute value of the angle of rotation of the body being isolated) is determined by Eq. (3.35), i.e.,

$$J_1(u_U^0) = \frac{\pi}{4} - \tan^{-1}\sqrt{\frac{\sigma - 1}{\sigma + 1}}, \quad \sigma = \sqrt{U}. \tag{3.36}$$

In this case, the functional $J_2(u)$ (the squared maximum load transmitted to the body) reaches its upper boundary

$$J_2(u_U^0) = U. \tag{3.37}$$

Introduce the function $g(U) = J_1(u_U^0)$. It follows from Eq. (3.36) that the function $g(U)$ is defined over the interval $[1, \infty)$. Also, it is continuous and monotonically decreases from $\pi/4$ to zero as U increases. Hence, there exists an inverse function $g^{-1}(D)$ which is defined over the interval $(0, \pi/4]$ and is continuous and monotonically decreasing on this interval. With reference to Theorem 1.1, we deduce that Problem 3.2 for $D \in (0, \pi/4)$ is the reciprocal of Problem 3.1. Using Theorem 1.1 we find that the minimum of the squared maximum transmitted load, $J_2(u_0^D)$, and the corresponding value of the maximum angle of rotation, $J_1(u_0^D)$, are given by

$$J_2(u_0^D) = g^{-1}(D) = \frac{(1 + \tan^2 D)^2}{4\tan^2 D}, \tag{3.38}$$

$$J_1(u_0^D) = D, \quad D \in (0, \pi/4),$$

while the optimal control $u_0^D(t)$ is defined by Eqs. (3.19) and (3.34) with $U = g^{-1}(D)$. It follows from Eq. (3.38) that $J_2(u_0^D) = 1$ for $D = \pi/4$, i.e. the maximum absolute value of the load reaches its lower boundary. Hence, the control $u_0^{\pi/4}(t)$ solves Problem 3.2 of limiting isolation capabilities for all $D \geq \pi/4$.

3.3 PARAMETRIC OPTIMIZATION OF SPRING AND DAMPER ISOLATORS.

Let us investigate the potential for protection of rotating objects from an instantaneous impact by using isolators containing a linear spring and a damper with a linear or quadratic characteristic. In this case, the motion of the object is governed by the initial-value problem of Eq. (3.1) where

$$M(\varphi, \dot{\varphi}, t) = -c|\dot{\varphi}|^r \mathrm{sign}(\dot{\varphi}) - k\varphi, \quad r = 1, 2, \tag{3.39}$$

where $c \geq 0$ and $k \geq 0$ are damping and stiffness factors, respectively.

Introduce the dimensionless (primed) variables

$$t' = |\beta|t, \quad c' = \frac{c|\beta|^{r-2}}{I}, \quad k' = \frac{k}{I\beta^2} \tag{3.40}$$

so that the initial value problem, which governs the system motion, is represented in the form of Eq. (3.11), where

$$u = -c'|\dot{\varphi}|^r \mathrm{sign}(\dot{\varphi}) - k'\varphi. \tag{3.41}$$

In what follows we omit the primes.

Consider Problem 3.2, with Y being the two-parameter set of isolator characteristics of Eq. (3.41), i.e.

$$Y = \{u(\varphi, \dot{\varphi}) : u = -c|\dot{\varphi}|^r \mathrm{sign}(\dot{\varphi}) - k\varphi, \quad c \geq 0, k \geq 0\}. \tag{3.42}$$

The damping (c) and stiffness (k) coefficients are the parameters (design variables) to be varied. The exponent r is fixed. The search for the solution of Problem 3.2 over the set of Eq. (3.42) is reduced to the determination of optimal parameters $c = c_0$ and $k = k_0$ that minimize the maximum absolute value of the acceleration (at an arbitrary point of the body being isolated), provided that the absolute value of the angle of rotation is constrained. For convenience, we will use the notation $\varphi(t, c, k)$, $J_1(c, k)$, and $J_2(c, k)$ instead of $\varphi(t; u)$, $J_1(u)$, and $J_2(u)$, respectively. In addition, introduce the notation

$$w^2(t, c, k) = u^2 + \dot{\varphi}^4 = \ddot{\varphi}^2(t, c, k) + \dot{\varphi}^4(t, c, k) \tag{3.43}$$

Accordingly,

$$\begin{aligned} J_1(c, k) &= \max\nolimits_{t\in[0,\infty)} |\varphi(t, c, k)|, \\ J_2(c, k) &= \max\nolimits_{t\in[0,\infty)} w^2(t, c, k). \end{aligned} \tag{3.44}$$

3.3.1.1 Proposition 3.2. If $D \geq 1$, then the optimal value of the damping coefficient is equal to zero ($c_0 = 0$), while the set of optimal stiffness coefficients is an interval given by

$$\{k_0\} = [1/D^2, 1]. \tag{3.45}$$

The maximum absolute value of the load reaches its lower boundary: $J_2(c_0, k_0) = 1$.

3.3.1.2 Proof. Solve the initial-value problem of Eq. (3.11) for u from the set of Eq.(3.42) under $c = 0$ and find

$$\varphi(t, 0, k) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}t) \tag{3.46}$$

$$w^2(t, 0, k) = \cos^4(\sqrt{k}t) - k\cos^2(\sqrt{k}t) + k \tag{3.47}$$

The right-hand side of Eq. (3.47) is a second-order polynomial in $\cos^2(\sqrt{k}t) \in [0, 1]$, with a positive coefficient of $\cos^4(\sqrt{k}t)$. Therefore, the maximum of the function of Eq. (3.47) is reached at $\cos^2(\sqrt{k}t) = 0$ or $\cos^2(\sqrt{k}t) = 1$. Hence,

$$J_2(0, k) = \max\{k, 1\}. \tag{3.48}$$

Equations (3.46) and (3.48) infer the proposition.

Note that for $D \geq 1$ the optimal value of $c_0 = 0$ is determined uniquely. Indeed, as follows from Eqs. (3.11), (3.42), and (3.43), for $c > 0$ we have $w^2(0, c, k) = c^2 + 1 > 1$ and, hence, $J_2(c, k_0) > J_2(0, k) = 1$.

Consider now the solution to the problem of optimizing the isolator characteristic parameters for $D < 1$ in the cases of $r = 1$ and $r = 2$. Figure 3.3 shows numerically constructed level curves of the function $J_2(c, k)$ for the linear isolator ($r = 1$). Qualitatively, for the isolator with a quadratic damper ($r = 2$), the curves are the same as for the linear isolator. The level curves of the function $J_2(c, k)$ are convex upward and the function $J_2(c, k)$ monotonically increases when moving away from the origin along any ray issuing from the origin and lying in the first quadrant ($c \geq 0, k \geq 0$) of the parameter plane (ck-plane). This implies that the function $J_2(c, k)$ has no internal extremum and that the optimal parameters cannot belong to coordinate axes $c = 0$ or $k = 0$. Hence, the optimal parameters belong to the curvilinear part of the admissible set boundary, i.e. to the curve

$$\gamma = \{c, k : J_1(c, k) = \max_t |\varphi(t, c, k)| = D, \quad c \geq 0, k \geq 0\} \tag{3.49}$$

Recall that the motion of the system in question is governed by the initial-value problem of Eq. (3.11) where u belongs to the set Y defined by Eq. (3.42). It coincides with (2.115) for $m = 1$ which, in turn, is a special case of the problem of Eq. (2.32), investigated in Section 2.3 of Chapter 2. The performance criterion $J_1(c, k)$ coincides with $I_1(c, k)$ defined by Eq. (2.33). As has been established in Section 2.3.2, the function $I_1(c, k)$ monotonically decreases with the growth of c and k (Proposition 2.3). This implies that the curve γ defined by Eq. (3.49) can be represented as a graph of a monotonically decreasing function $k = k(c)$ or $c = c(k)$.

For $r = 1$ and $r = 2$, we can obtain exact analytical equations of the curve γ. For $r = 1$, the function $J_1(c, k)$ is given by Eq. (1.172), and the equation of the curve γ is specified in an explicit form $J_1(c, k) = D$.

To obtain the equation of the portion of the state trajectory lying in the first quadrant for the case of $r = 2$, let us make use of Eq. (2.55). Specify $f(c, y) = cy^2$, $\varphi(k, x) = kx$, and $\beta = 1$ in Eq. (2.55) and then replace the variables x and y by φ and $\dot{\varphi}$, respectively. Thus we arrive at the initial-value problem

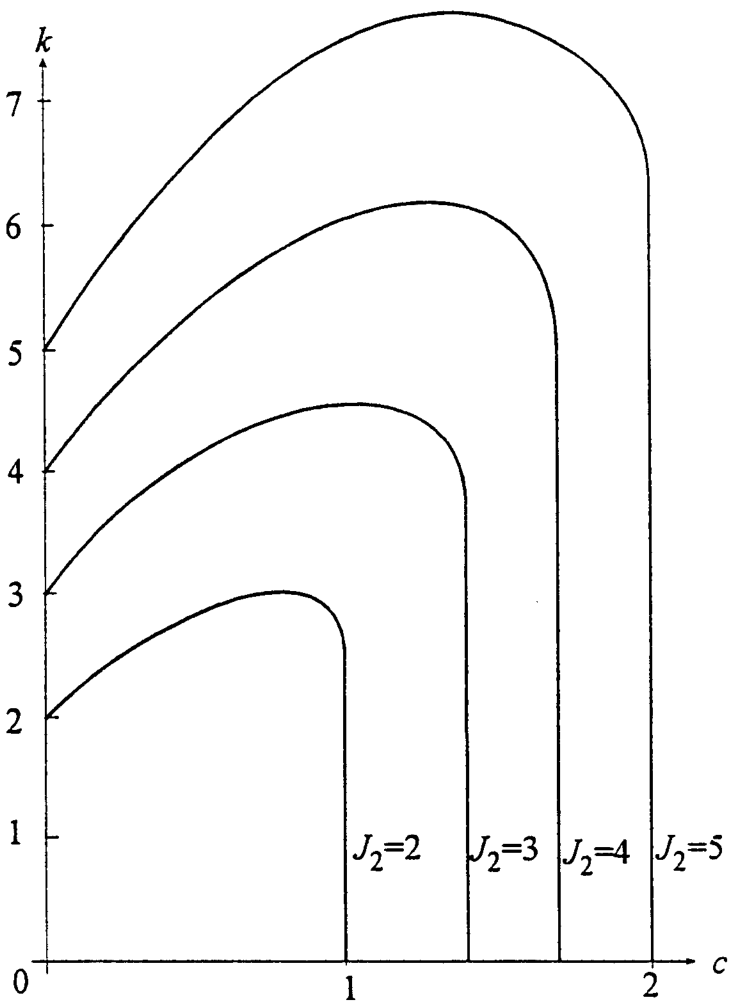


Figure 3-3. Level curves for the performance index J_2.

$$\dot{\varphi}\frac{d\dot{\varphi}}{d\varphi} = -c\dot{\varphi}^2 - k\varphi, \quad \dot{\varphi}(0) = 1 \tag{3.50}$$

or, equivalently,

$$\frac{1}{2}\frac{d\dot{\varphi}^2}{d\varphi} + c\dot{\varphi}^2 = -k\varphi, \quad \dot{\varphi}(0) = 1. \tag{3.51}$$

Equation (3.51) is a linear nonhomogeneous equation for $\dot{\varphi}^2$. Integration of this equation yields

$$\dot{\varphi}^2 = (1 - \frac{k}{2c^2})\exp(-2c\varphi) - \frac{k}{c}\varphi + \frac{k}{2c^2}. \tag{3.52}$$

According to Proposition 2.1, the maximum value of the variable φ is attained at the instant t_* at which the derivative $\dot{\varphi}$ vanishes for the first time. Hence,

$$J_1(c,k) = \varphi(t_*, c, k), \quad \dot{\varphi}(t_*, c, k) = 0. \tag{3.53}$$

Substituting Eq. (3.53) into Eq. (3.52) and taking into account that $J_1(c,k) = D$ on the curve γ, we obtain the equation specifying the curve γ in an explicit form

$$\left(1 - \frac{k}{2c^2}\right)\exp(-2cD) - \frac{k}{c}D + \frac{k}{2c^2} = 0. \tag{3.54}$$

Solve this equation for k to find

$$k = \frac{2c^2\exp(-2cD)}{2cD - 1 + \exp(-2cD)}. \tag{3.55}$$

Numerical calculation of the curves γ shows that they are convex downward.

The aforementioned properties of the curve γ and level curves of the criterion $J_2(c,k)$ imply that the function $J_2(c,k)$ has only one point of minimum on the curve γ. The problem of finding the minimum is reduced to a search for the minimum of a function of one variable. This can be accomplished numerically.

Figure 3.4 depicts the optimal parameters c_0 and k_0 versus D on the interval $0 < D \leq 1$, while Fig. 3.5 shows the optimal value $J_2(c_0, k_0)$ of the optimization criterion versus D. In these figures, the linear isolator with $r = 1$ is represented by solid lines, while the isolator with quadratic damping, corresponding to $r = 2$, by dashed lines.

The curves presented in Figs. 3.4 and 3.5, combined with Proposition 3.2, provide a complete solution for the problem of minimizing the maximum load transmitted to the body, subject to the constraint on the angle of rotation, by properly choosing the stiffness and damping coefficients of the isolator.

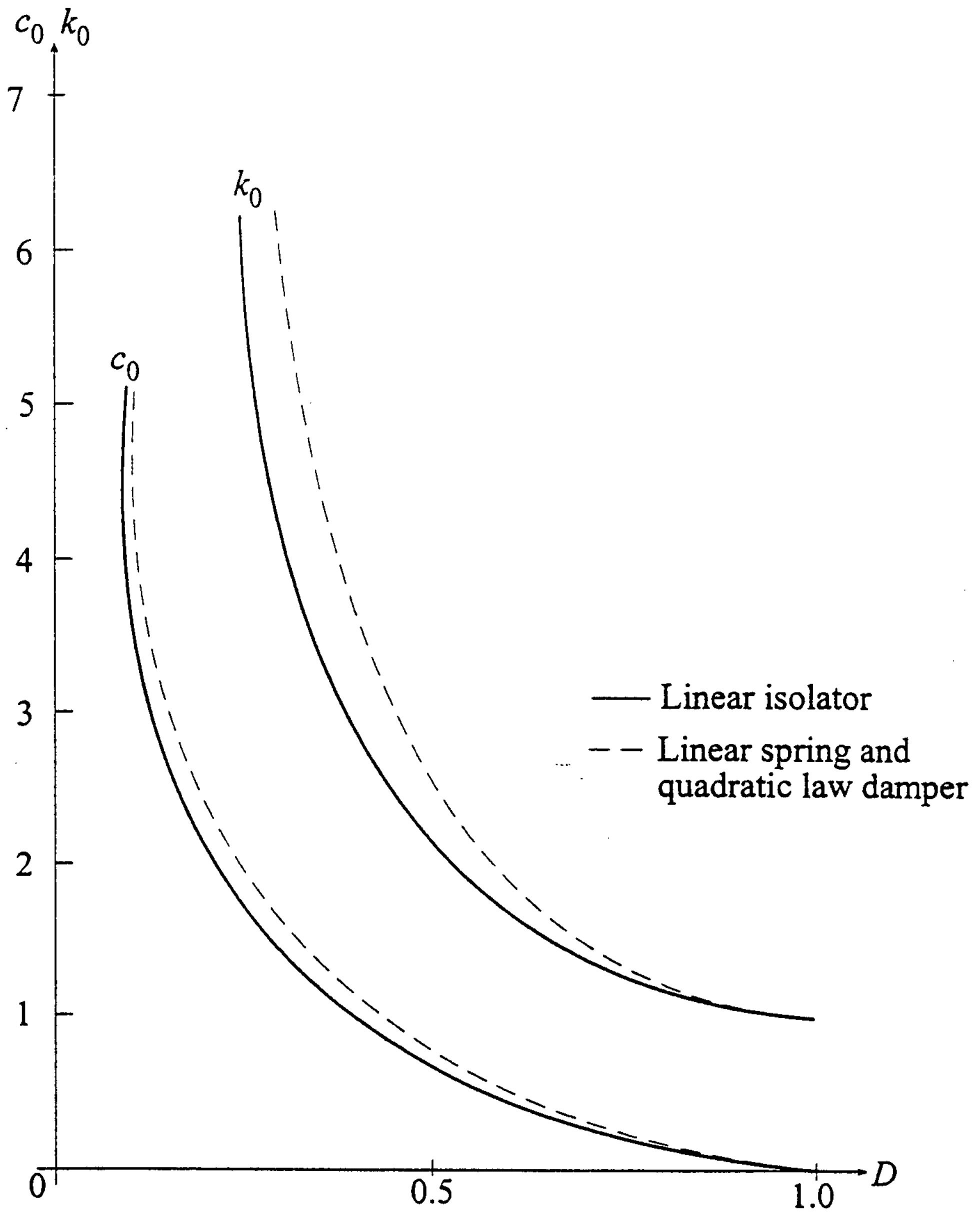


Figure 3-4. Dependence of optimal stiffness and damping coefficients on the maximum allowable angle of rotation.

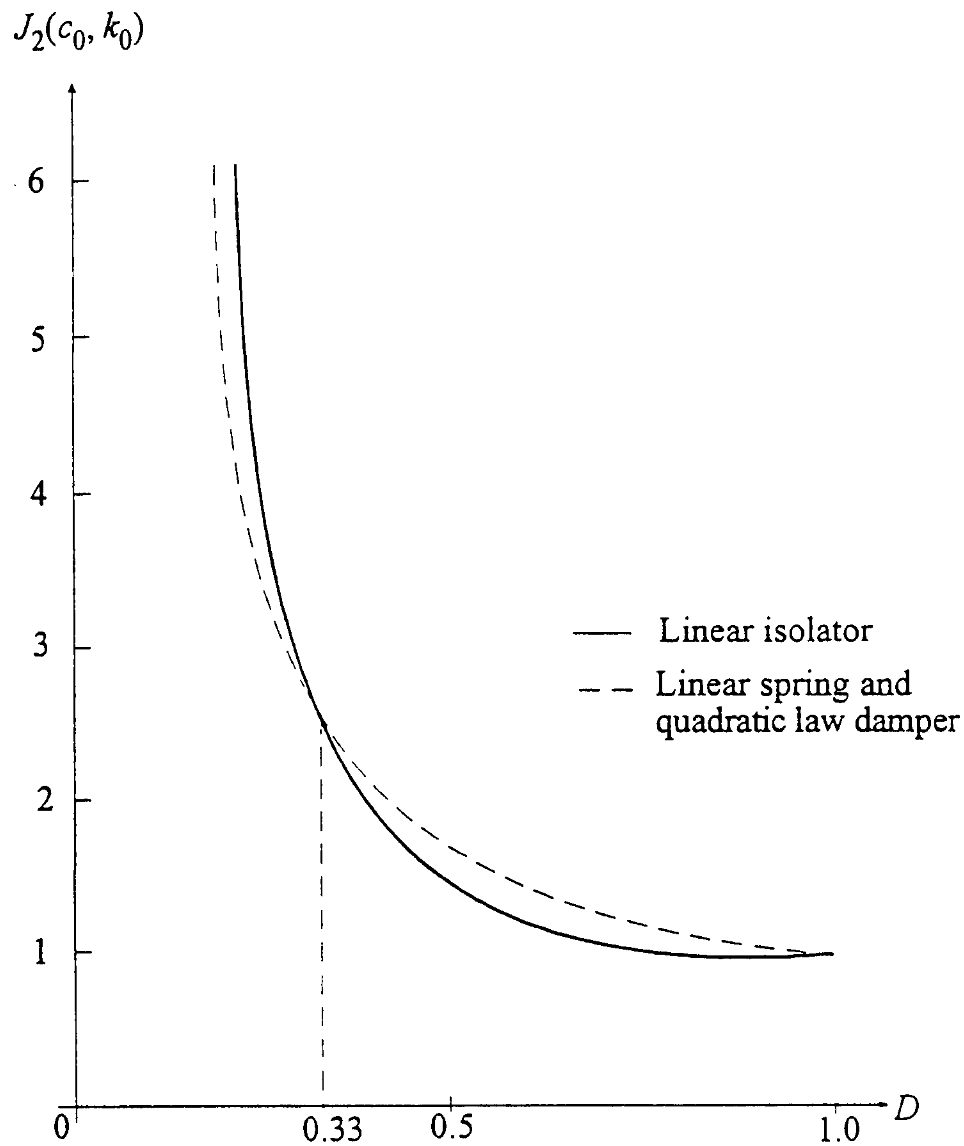


Figure 3-5. Dependence of optimal values of J_2 (c_0, k_0).

The optimal values of the optimization criterion J_2 monotonically decrease from ∞ to 1 as D increases from zero to unity, while the corresponding value of J_1 is equal to D. Hence, for the class of admissible characteristics defined by Eq. (3.42), Problem 3.1 is the reciprocal (in the sense of Theorem 1.1) of Problem 3.2. Given the maximum allowable value U of the criterion J_2 ($U \geq 1$), we can, using the curves of Fig. 3.5, determine the optimal value of the criterion J_1, desired in Problem 3.1, and, using the curves of Fig. 3.4, find the optimal damping (c^0) and stiffness (k^0) coefficients. As mentioned in Section 3.1, Problem 3.1 has no solution for $U < 1$.

3.4 DISCUSSION OF THE RESULTS.

In this chapter, we have considered problems of optimal isolation of a rotating body from an impulsive shock (instantaneous impact). Problems 3.1 and 3.2 for the optimal shock isolation of rotating objects differ from analogous problems for rectilinearly moving systems, considered in Chapter 2, in the presence of the term $\dot{\varphi}^4$ in Eq. (3.14) for the criterion characterizing the load transmitted to the body being isolated. This term describes a centripetal component of the rotating body acceleration. This leads to some distinctive features of the solutions to problems of rotating body optimal isolation as compared with solutions to isolation problems for rectilinearly moving systems. Unlike the case of a rectilinearly moving system, for a rotating object it is not possible to select at the outset the linear isolator or the isolator with a quadratic damper for all possible values of D (when solving Problem 3.2) or U (for Problem 3.1). As can be observed in Fig. 3.5, the isolator with a quadratic damper provides a lower load transmitted to the body being isolated for $D < 0.33$, whereas for $D > 0.33$, the linear isolator ensures better shock protection.

It is of interest to compare the shock isolation quality provided by optimal spring and damper isolators of the class of Eq. (3.42) with the limiting isolation capabilities established in Section 3.2. For this purpose, introduce the function

$$\eta(D) = \frac{\sqrt{J_2(c_0(D), k_0(D))} - \sqrt{J_2(u_0^D)}}{\sqrt{J_2(u_0^D)}} \tag{3.56}$$

showing the relative difference between the minimum value of the acceleration of points of a rotating body provided by an optimal spring and damper isolator and the absolute minimum of the acceleration reflecting the limiting isolation capabilities.

Figure 3.6 presents the graphs of the function $\eta(D)$ on the interval $0 < D \leq 1$ for the linear isolator (solid line) and the isolator with a linear spring and a quadratic damper (dashed line). The curves show that the spring-and-damper isolators investigated in Section 3.3 assure a rather high isolation quality differing from the limiting capabilities by no more than 4%.

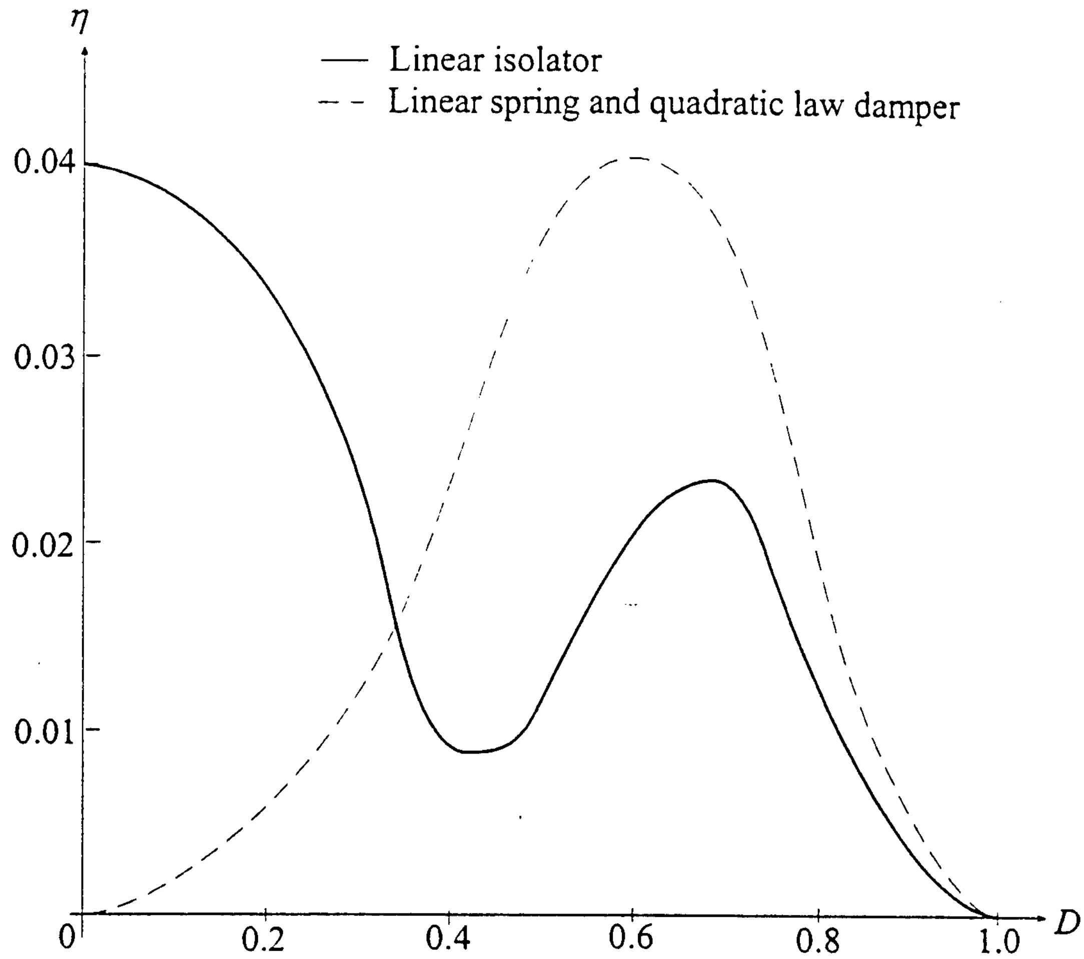


Figure 3-6. Relative difference between the limiting isolation capabilities and the isolation efficiency provided by passive isolators.

SECTION 4

OPTIMAL ISOLATION FOR A CLASS OF DISTURBANCES

4.1 STATEMENT OF THE PROBLEM.

In Chapter 2, when solving the problem of optimal shock protection, we assumed that the external disturbance is known in advance as a function of time. However, often information about the physical quantity characterizing the external disturbance is incomplete. Sometimes this uncertainty can be characterized as a class of external disturbances to which the actual disturbance can belong. For example, if a system with isolators undergoes an instantaneous impact (impulse) whose intensity cannot exceed S, the class of possible disturbances is the set of functions $F(t) = \beta\delta(t)$ subject to the constraint $|\beta| \leq S$. The problem of optimization of isolator characteristics when the external disturbances are incompletely prescribed can be formulated as a *minmax game problem.* The solution of this problem gives a guaranteed minimum of the optimization criterion which cannot be exceeded even under the most unfavorable disturbance of the class specified.

Let us formulate typical problems of optimization of characteristics of isolators allowing for a class of external disturbances. For the rectilinearly moving systems of Chapter 2, the motion of the body being isolated relative to the base is governed by the initial value problem

$$\ddot{x} + u(x, \dot{x}, t) = F(t), \quad x(t_0) = x^0, \quad \dot{x}(t_0) = \dot{x}^0, \tag{4.1}$$

where x is the displacement of the body relative to the base, $u(x, \dot{x}, t)$ is the isolator characteristic, $F(t)$ is the external disturbance, and x^0 and $\dot{x}^0$ are the initial values of the displacement and the velocity specified for some initial instant $t = t_0$. For the performance criteria, take the maximum value of the relative displacement of the body being isolated

$$J_1(u, F) = \max_{t \in [t_0, \infty)} |x(t)| \tag{4.2}$$

and the maximum force transmitted to the body (or the absolute acceleration of the body)

$$J_2(u, F) = \max_{t \in [t_0, \infty)} |u(x(t), \dot{x}(t), t)| \, . \tag{4.3}$$

Let the function $F(t)$ belong to a specified set Φ of possible disturbances. The set Φ should be determined with allowance for the system operating conditions and the designer's knowledge of the nature of the expected disturbances. In particular, the standard operating conditions to be allowed for when designing the isolation system can be established by a regulation organization, which, for example, is the case for the automotive industry.

4.1.1 Problem 4.1.

For the system governed by Eq. (4.1), find among a specified admissible set Y of isolator characteristics the optimal characteristic $u_0 \in Y$ such that

$$\max_{F\in\Phi} J_1(u_0, F) = \min_{u\in Y}\max_{F\in\Phi} J_1(u, F) \tag{4.4}$$

$$\max_{F\in\Phi} J_2(u_0, F) \leq U.$$

4.1.2 Problem 4.2.

For the system governed by Eq. (4.1), find among a specified admissible set Y of isolator characteristics the optimal characteristic $u^0 \in Y$ such that

$$\max_{F\in\Phi} J_2(u^0, F) = \min_{u\in Y}\max_{F\in\Phi} J_2(u, F) \tag{4.5}$$

$$\max_{F\in\Phi} J_1(u^0, F) \leq D.$$

Here, U and D are prescribed positive numbers indicating the maximum allowable values of the corresponding performance indices. Problems 4.1 and 4.2 generalize Problems 2.1 and 2.2 to the case of incomplete information about the external disturbances. Unlike Problems 2.1 and 2.2, in the present case, the initial conditions in Eq. (4.1) are assumed to be specified.

4.2 EXTERNAL DISTURBANCES SUBJECT TO AN INTEGRAL CONSTRAINT.

Let the body being isolated rest at the initial instant $t = 0$ at the position $x = 0$, i.e., at $t_0 = 0$, $x^0 = 0$ and $\dot{x}^0 = 0$ in Eq. (4.1). Consider a class Φ of external disturbances $F(t)$ specified as

$$\Phi = \{F(t) : \int_0^\infty |F(\tau)|\, d\tau \leq S\}, \tag{4.6}$$

where S is a prescribed number. Note that the impulse functions

$$F = \sum_i \beta_i \delta(t - t_i) \tag{4.7}$$

subject to the constraint $\sum_i |\beta_i| \leq S$ are included in this class. The index i runs through a finite or infinite set of consecutive positive integers. The functions of Eq. (4.7) describe a series of instantaneous impacts occurring at the instants t_i.

The integral in Eq. (4.6) has a clear physical interpretation in the case where the external force (or the acceleration of the base for a kinematical disturbance) does not change its direction. Then, the sign of the function $F(t)$ remains the same and the integral in Eq. (4.6) is equal to the absolute value of the impulse of the external force or to the absolute value of the change in the

velocity of the base. For the other cases, the integral in Eq. (4.6) has no simple interpretation. Recall that the *impulse* P *of a force* F *during a time interval* $t_1 \leq t \leq t_2$, is defined as

$$P = \int_{t_1}^{t_2} F(\tau)d\tau. \tag{4.8}$$

Let us present some important subclasses of the class Φ of Eq. (4.6).

4.2.0.1 Example 4.1. *Rectangular Pulse.* Consider a rectangular pulse

$$F(t) = \begin{cases} A, & \text{if } 0 \leq t \leq T \\ 0, & \text{if } t > T \end{cases}, \tag{4.9}$$

where A and T are specified constants. The function $F(t)$ is plotted in Fig. 4.1 for $A > 0$. The integral involved in Eq. (4.6) for function (4.9) is

$$\int_0^\infty |F(t)|dt = |A|T. \tag{4.10}$$

Thus, to satisfy the integral constraint of Eq. (4.6), the parameters A and T of the disturbance of (4.9) must satisfy the inequality

$$|A|T \leq S. \tag{4.11}$$

4.2.0.2 Example 4.2. *Half-Sine Pulse.* The half-sine pulse is defined as

$$F(t) = \begin{cases} A\sin\frac{\pi t}{T}, & \text{if } 0 \leq t \leq T \\ 0, & \text{if } t > T \end{cases}. \tag{4.12}$$

Function (4.12) is shown in Fig. 4.2 for $A > 0$. The integral of Eq. (4.6) for function (4.12) is

$$\int_0^\infty |F(t)|dt = \frac{2|A|T}{\pi}. \tag{4.13}$$

It follows from (4.13) that function (4.12) belongs to the class Φ of Eq. (4.6) if and only if the parameters A and T satisfy the inequality

$$|A|T \leq \frac{\pi}{2}S. \tag{4.14}$$

4.2.0.3 Example 4.3. *Exponential Pulse.* Define the exponential pulse as

$$F(t) = \frac{A}{T}t\exp\left(\frac{T-t}{T}\right), \quad 0 \leq t < \infty, \quad T > 0, \tag{4.15}$$

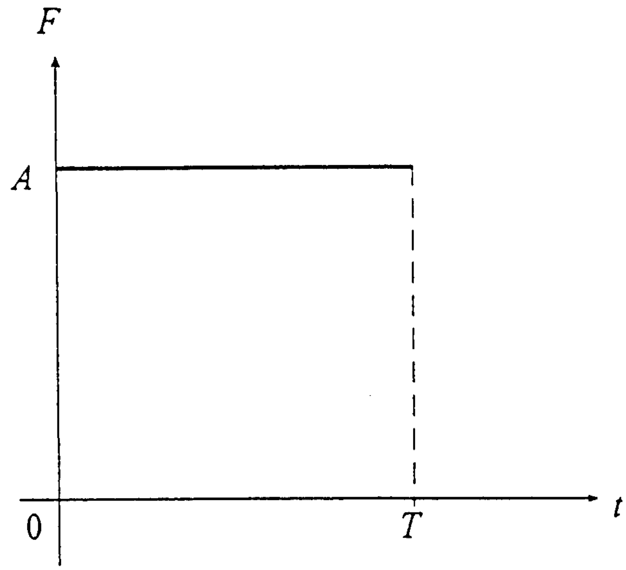


Figure 4-1. Rectangular pulse.

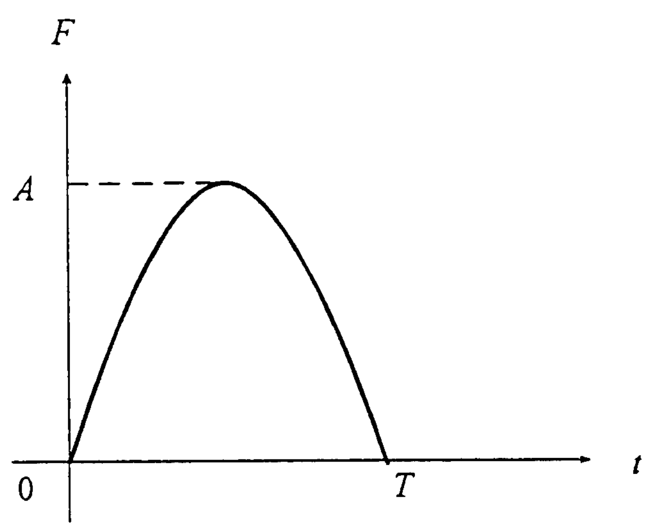


Figure 4-2. Half-sine pulse.

where A and T are specified constants. The function $F(t)$ is shown in Fig. 4.3 for $A > 0$. Unlike the rectangular and half-sine pulses, the exponential pulse does not vanish for any $t > 0$. The integral of Eq. (4.6) for function (4.15) is

$$\int_0^\infty |F(t)|dt = |A|T\exp(1). \tag{4.16}$$

It follows from (4.16) that function (4.15) belongs to the class Φ of Eq. (4.6) if and only if the parameters A and T satisfy the inequality

$$|A|T \leq S\exp(-1). \tag{4.17}$$

4.2.0.4 Example 4.4. *Decaying Sinusoidal Disturbance.* Consider a disturbance of the form

$$F(t) = A\exp\left(-\frac{t}{T_1}\right)\sin\left(\frac{2\pi}{T_2}t\right), \quad 0 \leq t < \infty, \tag{4.18}$$

$$T_1 > 0, \quad T_2 > 0.$$

The plot of this function is presented in Fig. 4.4 for $A > 0$. Unlike the functions $F(t)$ in Examples 4.1, 4.2, and 4.3, function (4.18) is variable in sign. Specifically, if $A > 0$, then

$$F(t) > 0, \quad \text{if} \quad T_2(n-1) < t < T_2(n-\frac{1}{2}), \quad n = 1, 2, \ldots \tag{4.19}$$

$$F(t) < 0, \quad \text{if} \quad T_2(n-\frac{1}{2}) < t < T_2 n, \quad n = 1, 2, \ldots$$

The integral of Eq. (4.6) for function (4.18) is

$$\int_0^\infty |F(t)|dt = |A|\int_0^\infty \exp\left(-\frac{t}{T_1}\right)\left|\sin\left(\frac{2\pi}{T_2}t\right)\right|dt = \tag{4.20}$$

$$\frac{2\pi|A|T_2}{4\pi^2 + (T_2/T_1)^2}\coth\left(\frac{T_2}{4T_1}\right).$$

It follows from (4.20) that for function (4.18) to belong to the class Φ of Eq. (4.6), the parameters A, T_1, and T_2 must satisfy the inequality

$$\frac{2\pi|A|T_2}{4\pi^2 + (T_2/T_1)^2}\coth\left(\frac{T_2}{4T_1}\right) \leq S. \tag{4.21}$$

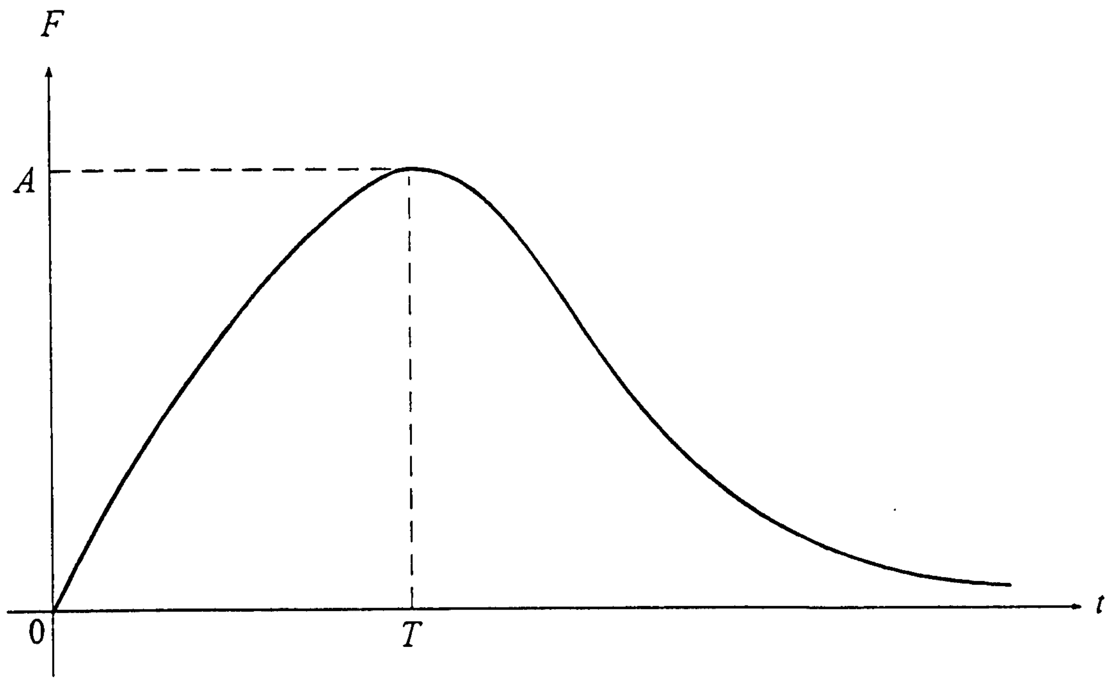


Figure 4-3. Exponential pulse.

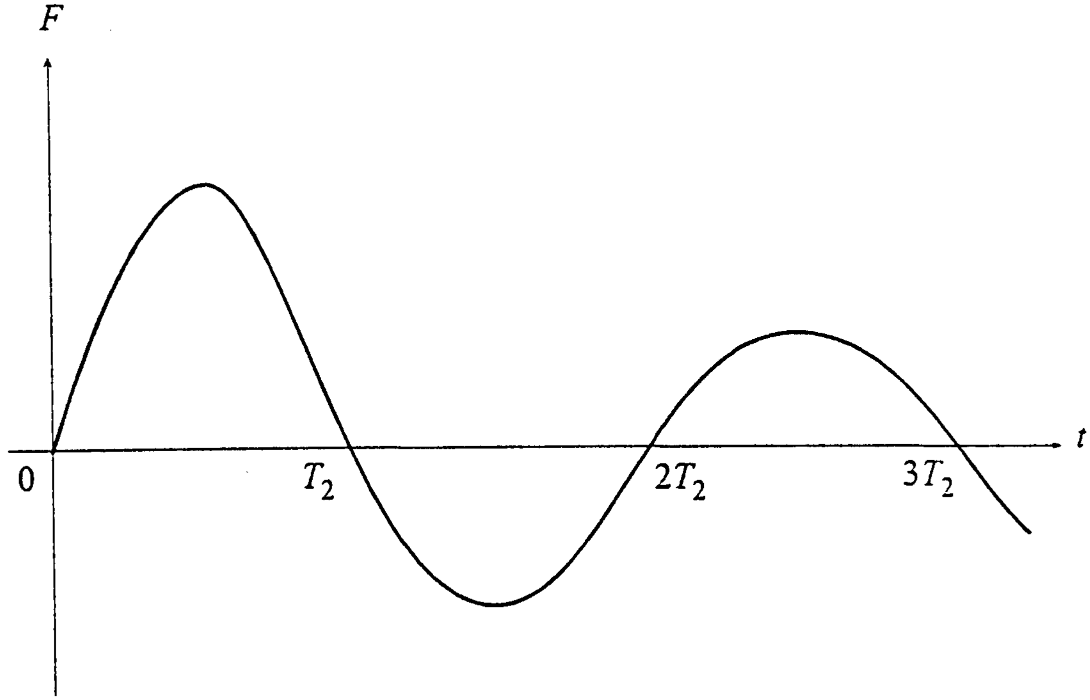


Figure 4-4. Decaying sinusoidal disturbance.

As the class Y of admissible isolator characteristics, we consider piecewise continuous functions $u(x, \dot{x}, t)$. Sometimes, additional conditions will be imposed on the admissible characteristics of the isolator and, hence, a more narrow class Y will be treated.

4.2.1 Worst Disturbance Problem.

4.2.1.1 Problem Formulation. Prior to addressing the solution of the isolator design Problems 4.1 and 4.2, we will consider auxiliary problems of determining disturbances which maximize functionals $J_1(u, F)$ and $J_2(u, F)$ for a prescribed characteristic u.

4.2.1.2 Problem 4.3. Find $F_1^0 \in \Phi$ such that $J_1(u, F_1^0) = \max\limits_{F \in \Phi} J_1(u, F)$.

4.2.1.3 Problem 4.4. Find $F_2^0 \in \Phi$ such that $J_2(u, F_2^0) = \max\limits_{F \in \Phi} J_2(u, F)$.

Here, Φ is the class of integrally constrained functions defined in Eq. (4.6). Sometimes. Problems 4.3 and 4.4 are called *worst disturbance problems* (with respect to the corresponding performance indices).

The solution to Problems 4.1 and 4.2 would be significantly simplified if the worst disturbance were the same for both functionals J_1 and J_2, i.e., $F_1^0 = F_2^0 = F^0$, and, moreover, did not depend on the isolator characteristic. In this case, the search for the optimal characteristic would be reduced to the solution of the problem on the optimal isolator for a specified external disturbance that is the worst disturbance. Unfortunately, these conditions are not met in the general case. However, for some classes of isolator characteristics, including those widely used in practice, these conditions are met.

4.2.1.4 Isolators with Linear and Coulomb Damper. Consider a parametric family of passive isolators with the characteristics

$$u(x, \dot{x}; h, c, k) = hq + c\dot{x} + \varphi(k, x)\text{sign}(x), \tag{4.22}$$

$$q = \begin{cases} \text{sign}(\dot{x}), & \text{for } \dot{x} \neq 0 \\ \nu/h, & \text{for } \dot{x} = 0 \quad \text{and} \quad |\nu| \leq h \\ \text{sign}(\nu), & \text{for } \dot{x} = 0 \quad \text{and} \quad |\nu| > h \end{cases},$$

$$\nu = F - \varphi(k, x)\text{sign}(x),$$

where the terms hq, $c\dot{x}$, and $\varphi(k, x)\text{sign}(x)$ characterize the dry friction, viscous linear friction, and elasticity of the isolator. The quantities h, c, and k are non-negative numbers, where h is the dry-friction factor, c is the viscous-friction coefficient, and k is the stiffness factor. The function $\varphi(k, x)$ is non-negative, even in x, and nondecreasing for $x > 0$. The function $\varphi(k, x)$ describes the characteristic of a stiffness element. For the dry friction we use the Coulomb model described in Section 2.3.7. In what follows, we use the terms dry friction, Coulomb friction, and Coulomb damping as synonyms.

The class of characteristics of Eq. (4.22) covers virtually all stiffness elements used in practice. The only dampers allowed by this class are linear and Coulomb damping elements.

We will prove that *for any isolator characteristic of the class of Eq. (4.22), the solution to both Problems 4.3 and 4.4 for the disturbance class* Φ *of Eq. (4.6) is given by the same function*

$$F_1^0 = F_2^0 = \pm S\delta(t). \tag{4.23}$$

That is, the worst disturbance is an instantaneous impact of maximum allowable intensity. This reduces Problems 4.1 and 4.2 of the optimal isolation for the class of integrally constrained external disturbances to the corresponding problems of protection against an instantaneous impact of a specified intensity. Methods of solving such problems have been considered in detail in Chapter 2.

We will show also that the parametric family specified by Eq. (4.22) contains a characteristic that solves Problems 4.1 and 4.2 for the class of all piecewise continuous functions $u(x, \dot{x})$, i.e., a characteristic implementing the limiting isolation capabilities.

4.2.1.5 Solution of the Worst Disturbance Problem for Isolators with Linear and Coulomb Friction:

4.2.1.6 Preliminary Analysis. Substitute $u(x, \dot{x}, t)$ of Eq. (4.22) into (4.1) and set $t_0 = 0, x^0 = 0$, and $\dot{x}^0 = 0$

$$\begin{gathered} \ddot{x} + hq + c\dot{x} + \varphi(k, x)\text{sign}(x) = F(t), \\ x(0) = 0, \quad \dot{x}(0) = 0. \end{gathered} \tag{4.24}$$

Denote the solution of this initial-value problem by $x(t; F)$.

4.2.1.7 Lemma 4.1. The inequality

$$|\dot{x}(t; F)| \leq S, \tag{4.25}$$

where S is defined in Eq. (4.6), is satisfied for any $F \in \Phi$ and $t \geq 0$.

4.2.1.8 Proof. Introduce the function

$$W(x, \dot{x}) = \frac{\dot{x}^2}{2} + \int_0^x \varphi(k, \xi)\text{sign}(\xi)d\xi, \tag{4.26}$$

which is the total mechanical energy of the system, and the function $\Psi(t; F) = W[x(t; F), \dot{x}(t; F)]$ which defines the time history of the energy along the solution of the problem of Eq. (4.24). Note that the integral (the potential energy) in Eq. (4.26) is nonnegative, since the function $\varphi(k, x)$ is assumed to be non-negative. Therefore,

$$\Psi(t;F) \geq \frac{\dot{x}^2(t;F)}{2}, \quad \Psi(0;F) = 0. \tag{4.27}$$

The latter equality follows from Eq. (4.26) and the initial conditions in Eq. (4.24).

Differentiating the energy along the trajectories of the system of Eq. (4.24) we obtain

$$\frac{d\Psi(t;F)}{dt} = -[hq + c\dot{x}(t;F)]\dot{x}(t;F) + \dot{x}(t;F)F(t).$$

The first term on the right-hand side describes the energy dissipation due to Coulomb and linear friction and is nonpositive. Hence,

$$\frac{d\Psi(t;F)}{dt} \leq \dot{x}(t;F)F(t) \leq |\dot{x}(t;F)||F(t)|. \tag{4.28}$$

From Eqs. (4.27) and (4.28) it follows that

$$\frac{d\Psi(t;F)}{dt} \leq \sqrt{2\Psi(t;F)}|F(t)|. \tag{4.29}$$

Dividing Eq. (4.29) by $\sqrt{2\Psi(t;F)}$ and integrating with respect to time from 0 to t we obtain

$$\sqrt{2\Psi(t;F)} \leq \int_0^t |F(\tau)|d\tau \leq \int_0^\infty |F(\tau)|d\tau \leq S. \tag{4.30}$$

Equation (4.25) follows from these relations and the inequality in Eq. (4.27).

This completes the proof of the lemma.

Let us reduce Eq. (4.24) to a standard form of a system of first-order differential equations

$$\begin{aligned} &\dot{x} = y \\ &\dot{y} = -hq - cy - \varphi(k,x)\mathrm{sign}(x) + F(t) \\ &x(0) = 0, \quad y(0) = 0. \end{aligned} \tag{4.31}$$

Denote by $x(t;F)$ and $y(t;F)$ the solution of the initial-value problem of Eq. (4.31).

Consider first the case of an instantaneous impact $F(t) = \beta\delta(t)$ with $|\beta| \leq S$. In this case, the initial-value problem of Eq. (4.31) is represented as

$$\begin{aligned} &\dot{x} = y \\ &\dot{y} = -hq - cy - \varphi(k,x)\mathrm{sign}(x) \\ &x(0) = 0, \quad y(0) = \beta. \end{aligned} \tag{4.32}$$

For $\beta = S$ and $\beta = -S$, consider the motion of the system until the instant at which the velocity y vanishes for the first time. The corresponding segments of the phase trajectories lie in the first

and the third quadrants, respectively, of the phase plane xy (Fig. 4.5). Denote by γ^+ the curve corresponding to $\beta = S$ and by γ^-, the curve for $\beta = -S$. Since $x(t)$ changes monotonically on the time interval under consideration, the curves γ^+ and γ^- can be represented by functions $y = y_0^+(x)$ and $y = y_0^-(x)$. These functions are governed by the first-order differential equation

$$\frac{dy}{dx} = -\frac{h\text{sign}(y) + cy + \varphi(k,x)\text{sign}(x)}{y} \tag{4.33}$$

with initial conditions $y(0) = S$ (for $y_0^+(x)$) or $y(0) = -S$ (for $y_0^-(x)$). Equation (4.33) is obtained by dividing the second relation of Eq. (4.32) by the first one, taking into account the expression for q in Eq. (4.22).

Denote by A the starting point of the curve γ^+ ($x_A = 0, \quad y_A = S$), by B the terminal point of the curve $\gamma^+(x_B, \quad y_B = y_0^+(x_B) = 0)$, by A' the starting point of the curve γ^- ($x_{A'} = 0, \quad y_{A'} = -S$), and by B' the terminal point of the curve $\gamma^-(x_{B'}, \quad y_{B'} = y_0^-(x_{B'}) = 0)$. Let us consider a figure $ABCA'B'C'$ on the phase plane xy shown shaded in Fig. 4.6. This figure is bounded by the curves γ^+ and γ^- and straight segments $BC, CA', B'C'$, and $C'A$. The coordinates of the points C and C' are given by $x_C = x_B, \quad y_C = y_{A'}, \quad x_{C'} = x_{B'}, \quad y_{C'} = y_A$. Therefore, quadrangles $OBCA'$ and $OB'C'A$ are rectangles. Since Eq. (4.33) is invariant with respect to the simultaneous change of x for $-x$ and y for $-y$, the figure $ABCA'B'C'$ is symmetrical with respect to the coordinate origin. We denote the shaded region in Fig. 4.6 by μ^*.

4.2.1.9 Basic Lemma:

4.2.1.10 Lemma 4.2. Phase trajectories of the system of Eq. (4.31) do not leave the region μ^* for any $F \in \Phi$, where the class Φ is defined by Eq. (4.6).

4.2.1.11 Proof. First, prove this lemma for impact disturbances of the form $F = \beta\delta(t)$ with $|\beta| < S$. Such disturbances belong to the class of Eq. (4.6). The phase trajectories corresponding to $|\beta| < S$ issue from the points of the segment AA', i.e., belong to the region μ^*. To leave this region, the phase trajectory must intersect the region boundary. No phase trajectory can intersect either curve γ^- or the curve γ^+, since these curves are phase trajectories of the system of Eq. (4.32), and phase trajectories of the systems whose right-hand sides do not depend on time cannot intersect each other. No phase trajectory can intersect the segments AC or $A'C'$, because in this case the absolute value of the velocity of the body being isolated would become greater then S, which is impossible for $|\beta| < S$, according to Lemma 4.1. Finally, the phase trajectories cannot intersect the "vertical" segments $B'C'$ and BC, because, otherwise, the coordinate x would increase for $y < 0$ or decrease for $y > 0$. Such a behavior of the phase trajectory is impossible, since $y = \dot{x}$ and x increases (decreases) only if $y > 0$ ($y < 0$).

Now, prove the lemma for the general case. Unlike the case of an impact disturbance, in the general case the right-hand side of the system of Eq. (4.31) depends on time. Therefore, different phase trajectories may intersect each other and, moreover, the phase trajectories may have multiple points (points of self-intersection). This substantially complicates the investigation, as compared with the case of the impact disturbance.

For simplicity we assume that $F(t)$ is a piecewise continuous function. In this case, phase trajectories are continuous. If the external disturbance contains impulses applied at certain

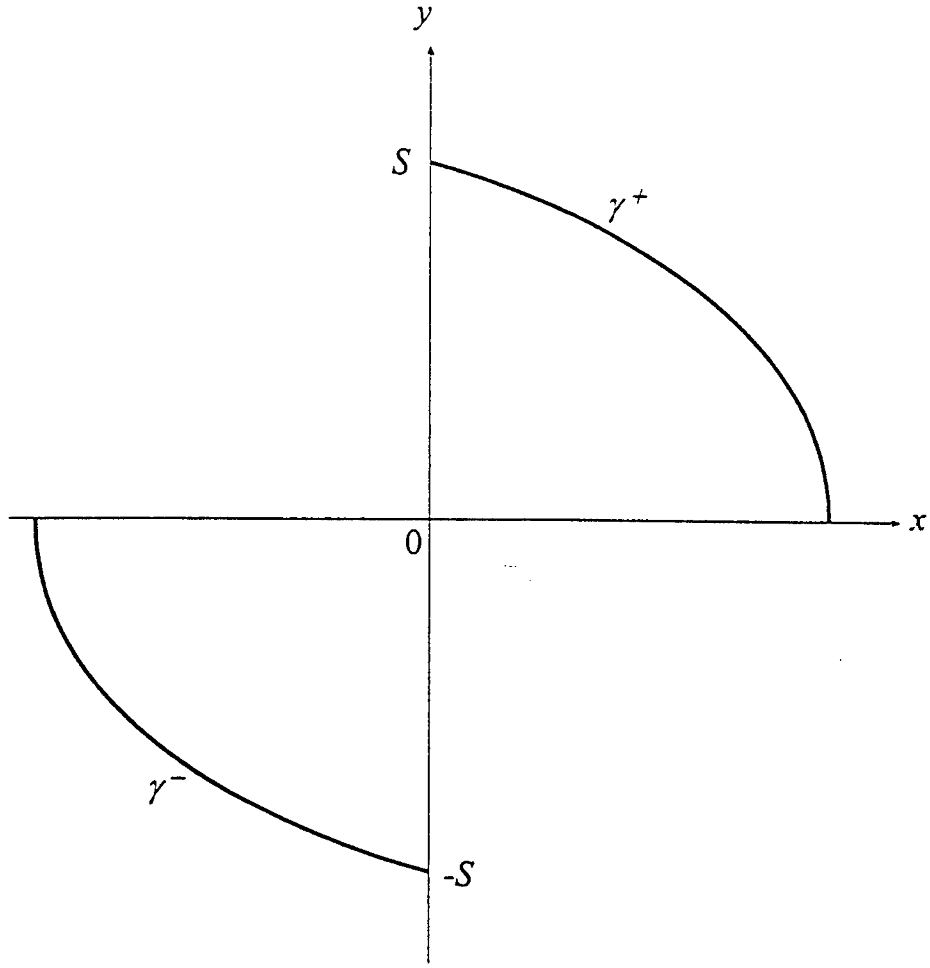


Figure 4-5. Curves γ^+ and γ^-.

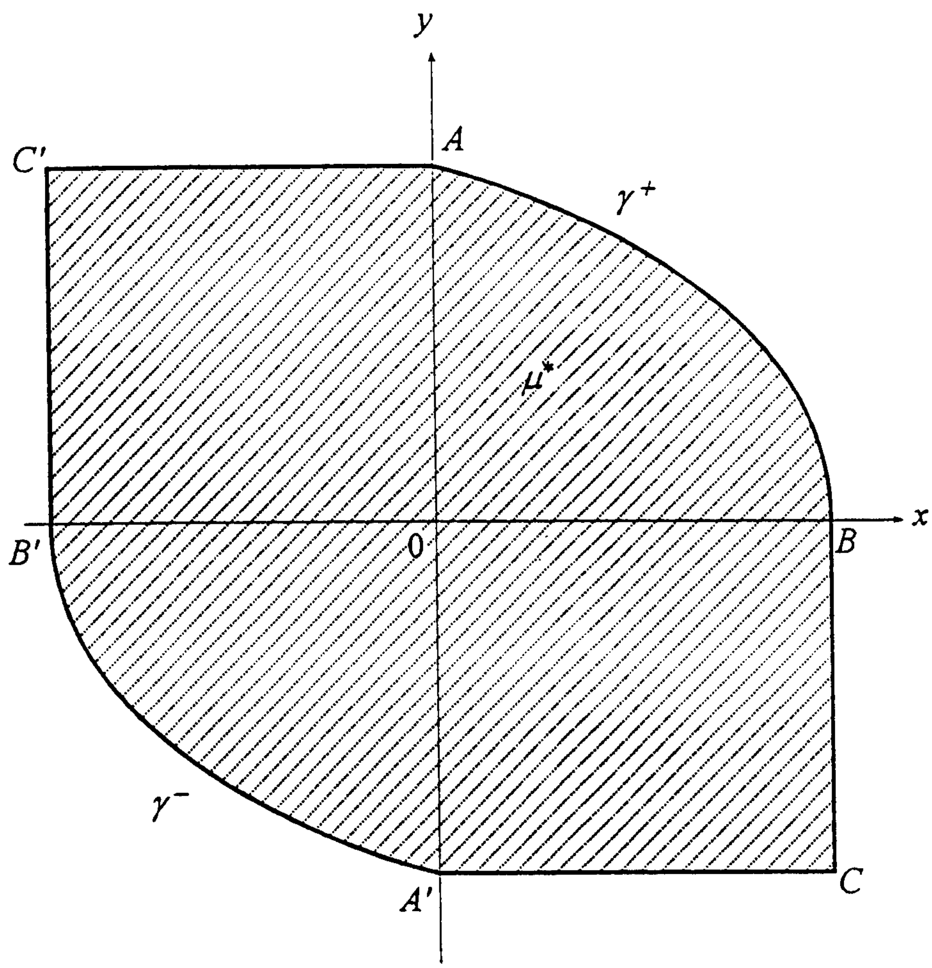


Figure 4-6. Region μ^*.

instants, the velocity (and hence, the phase trajectory) experiences jumps at these instants. This somewhat complicates the proof. However, the basic arguments are retained. It is worth noting here that an impulse function can be treated as a limit (in a certain sense) of a sequence of piecewise continuous functions.

It is easy to show that the representative point of the system of Eq. (4.31) cannot leave the region μ^* through the straight segments BC, CA', $B'C'$ or $C'A$. The intersection of vertical segments BC and $B'C'$ from inside the region μ^* is impossible because x monotonically increases in time for $y = \dot{x} > 0$ and monotonically decreases for $y < 0$. The intersection of horizontal segments $A'C$ and AC' is impossible according to Lemma 4.1.

Thus, it remains to prove that the representative point cannot leave the region μ^* by intersecting curves γ^+or γ^-. The initial-value problem of Eq. (4.31) is invariant to the simultaneous change in the sign of the variables x, y, and F. Moreover, $-F \in \Phi$ if $F \in \Phi$. Therefore, it is sufficient to prove this proposition for one of the curves γ^+or γ^-.

Let us consider the behavior of the phase trajectories of the system of Eq. (4.31) in the first quadrant of the xy plane. The representative point can enter the first quadrant by intersecting either x-or y-axes and can leave this quadrant by intersecting the x-axis. Denote by t_i^- and t_i^+ $(i = 1, 2, ...,)$ the instants of the ith intrusion into and the ith departure from the first quadrant by the representative point of the system. During the interval $[t_i^-\ ,\ t_i^+\ ]$, the representative point moves in the first quadrant drawing a segment of the phase trajectory which we denote by γ_i. Introduce the notation

$$x_i^- = x(t_i^-; F), \quad x_i^+ = x(t_i^+; F), \quad y_i^- = y(t_i^-; F), \quad y_i^+ = y(t_i^+; F). \tag{4.34}$$

As has been mentioned, the representative point can leave the first quadrant only by intersecting the x-axis, and, hence, $y_i^+ = 0$. Since $\dot{x} = y(t; F) > 0$ for $t \in (t_i^-, t_i^+)$, the coordinate x monotonically increases on this interval, and the curve γ_i can be represented as a function $y = y_i(x)$ defined for $x \in [x_i^-, x_i^+]$. Denote by Γ_0 the set of curves γ_i beginning on the y-axis, i.e. $x_i^- = 0$ and $y_i^- \geq 0$ for $\gamma_i \in \Gamma_0$. Note that the set Γ_0 is non-empty, if the set of all γ_i is non-empty. Indeed, according to the initial conditions in Eq. (4.31), the representative point of the system coincides with the coordinate origin at $t = 0$ (and, hence, belongs to the y-axis). The increase of x for $y > 0$ and the decrease for $y < 0$ imply that the representative point can enter the first quadrant for the first time immediately after the beginning of the motion or by intersecting the y-axis from the second quadrant. Thus, $\gamma_1 \in \Gamma_0$.

Consider the curves γ_j and γ_m, $j < m$, such that γ_j and γ_m belong to Γ_0, whereas γ_s with $s = j + 1, ..., m - 1$ does not belong to Γ_0. Note the number m depends on j. Sometimes, we will use notation $m(j)$, if the omission of the argument can lead to confusion. We will perform some preliminary constructions. Consider first the curves γ_j and γ_{j+1}. These curves may or may not have a common point. If the former case occurs, we denote the coordinates of the first common point by x_{j+1}^j and y_{j+1}^j and form the curve H_j^1 consisting of the segment of the curve γ_j from its beginning to the first intersection point with γ_{j+1} and the segment of γ_{j+1} from the intersection point to the end of the curve γ_{j+1} (Fig. 4.7). Analytically, the curve H_j^1 is represented by

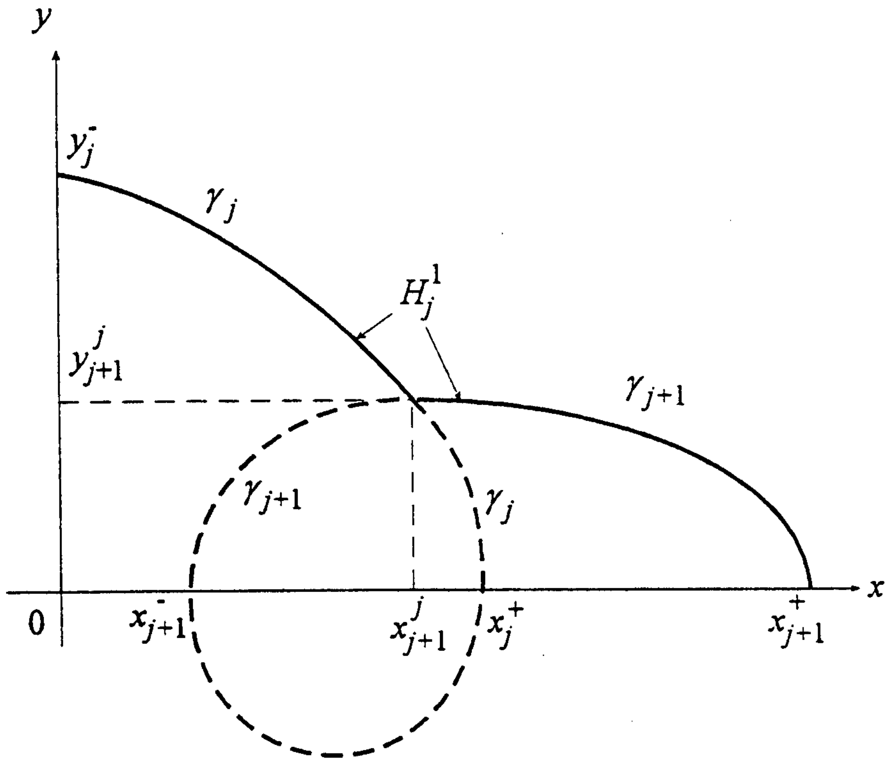


Figure 4-7. Curve H_j^1 consisting of parts of the curves γ_j and γ_{j+1}.

$$y = \bar{y}_j^1(x) = \begin{cases} y_j(x), & \text{for} \quad x \in [0, x_{j+1}^j] \\ y_{j+1}(x), & \text{for} \quad x \in (x_{j+1}^j, x_{j+1}^+] \end{cases} \tag{4.35}$$

Denote by ξ_{j+1}^j and ζ_{j+1}^j the instants of passing through the intersection point (x_{j+1}^j, y_{j-1}^j) by the curves γ_j and γ_{j+1}, respectively. Accordingly,

$$\begin{gathered} \xi_{j+1}^j \in [t_j^-, t_j^+], \quad \zeta_{j+1}^j \in [t_{j+1}^-, t_{j+1}^+], \\ x(\xi_{j+1}^j; F) = x(\zeta_{j+1}^j; F) = x_{j+1}^j, \\ y(\xi_{j+1}^j; F) = y(\zeta_{j+1}^j; F) = y_{j+1}^j. \end{gathered} \tag{4.36}$$

If the curves γ_j and γ_{j+1} have no common points (in this case, the curve γ_{j+1} lies between the curve γ_j and the x-axis as shown in Fig. 4.8), we will assume that the curve H_j^1 coincides with γ_j, i.e.,

$$\bar{y}_j^1(x) = y_j(x), \quad x \in [0, x_j^+] \tag{4.37}$$

In a similar manner, considering the curves H_j^1 and γ_{j+2}, we can construct the curve H_j^2, and so on. The general procedure of constructing the curve H_j^{r+1} when having constructed H_j^r, where $r = 1, \ldots, m - j - 1$ is as follows.

Let the curve γ_l for some $l = j + 2, \ldots, m - 1$ intersect the curve H_j^r. Denote the coordinates of the first (after $t = t_l^-$) point of intersection of these curves by x_l^j and y_l^j. The curve H_j^{r+1} is composed of the segment of the curve H_j^r (from its beginning to the point (x_l^j, y_l^j)) and the segment of the curve γ_l from the intersection point to the end point. Thus, the curve H_j^{r+1} can be represented as

$$y = \bar{y}_j^{r+1}(x) = \begin{cases} \bar{y}_j^r(x), & x \in [0, x_l^j] \\ y_l(x), & x \in (x_l^j, x_l^+] \end{cases} . \tag{4.38}$$

If the curves H_j^r and γ_l do not intersect, then we take the curve H_j^{r+1} that coincides with H_j^r, i.e.,

$$\bar{y}_j^{r+1}(x) = \bar{y}_j^r(x). \tag{4.39}$$

Introduce the set G_0 of curves H_j^p containing all curves $\gamma_j \in \Gamma_0$ and all curves H_j^r constructed for the curves γ_j according to the above procedure. For convenience, denote $H_j^0 = \gamma_j$ (accordingly, $\bar{y}_j^0(x) = y_j(x)$), so that the subscript p ranges from 0 to $m(j) - j - 1$. With each curve $H_j^p \in \Gamma_0$, we associate the set Θ_j^p of phase points bounded by the curve H_j^p and the x- and y- axes (see Fig. 4.9). According to the definition of the set G_0, for any γ_i, there exists a curve H_j^p such that $\gamma_i \in \Theta_j^p$. Thus, to prove that the phase trajectory of the system of Eq. (4.31) cannot leave the set μ^* immediately from the first quadrant, it is sufficient to show that curves $H_j^p \in G_0$ do not intersect the curve γ^+.

Let us consider a curve H_j^p. For simplicity we assume that this curve consists of segments of only two curves, γ_j and γ_n. Let x_n^j and y_n^j be the coordinates of the point of intersection of the curves

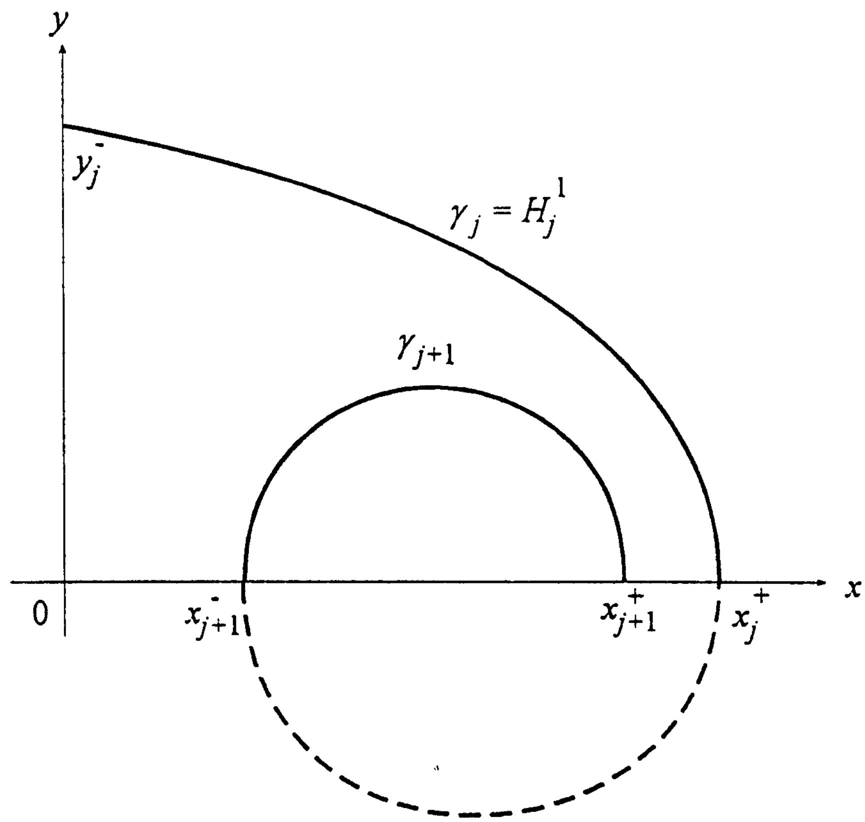


Figure 4-8. Curve H_j^1 coinciding with the curve γ_j.

γ_j and γ_n, while ξ_n^j and ζ_n^j ($\xi_n^j < \zeta_n^j$) be the instants of passing through this point by the curves γ_j and γ_n, respectively. Consider the external disturbance

$$F^*(t) = \begin{cases} F(t), & \text{for} \quad t \in [0, \xi_n^j) \\ F(t - \xi_n^j + \zeta_n^j), & \text{for} \quad t \geq \xi_n^j \end{cases} \quad . \tag{4.40}$$

The disturbance $F^*(t)$ belongs to the class Φ of Eq. (4.6) if $F(t) \in \Phi$. The function provides the motion of the representative point along the curve H_j^p on the time interval $[t_j^-, t_n^+ + \xi_n^j - \zeta_n^j]$. Since $y(t; F^*) > 0$ on the interval $(t_j^-, t_n^+ + \xi_n^j - \zeta_n^j)$ and, hence, $x(t; F^*)$ monotonically increases, one can parametrize the phase trajectory H_j^p and the time of motion along this trajectory by the coordinate x :

$$y = \bar{y}_j^p(x), \quad t = t_j^p(x), \quad x \in [0, x_n^+] \tag{4.41}$$

This parametrization induces the parametrization of the disturbance $F^*(t)$;

$$F^*(t_j^p(x)) = F_j^p(x), \quad x \in [0, x_n^+]. \tag{4.42}$$

The function $\bar{y}_j^p(x)$ is the solution to the initial-value problem

$$\frac{dy}{dx} = -\frac{h + cy + \varphi(k, x) - F_j^p(x)}{y}, \quad y(0) = y_j^-. \tag{4.43}$$

The differential equation in Eq. (4.43) is obtained by dividing the second relation in Eq. (4.31) by the first relation and replacing $F(t)$ by $F_j^p(x)$. The initial-value problem of Eq. (4.43) can be represented in the integral form

$$\bar{y}_j^p(x) = y_j^- - \int_0^x z[\xi, \bar{y}_j^p(\xi)]d\xi + \int_0^x \frac{F_j^p(\xi)}{\bar{y}_j^p(\xi)}d\xi, \tag{4.44}$$

$$z[x, \bar{y}_j^p(x)] = \frac{h + c\bar{y}_j^p(x) + \varphi(k, x)}{\bar{y}_j^p(x)}.$$

The following estimate is valid for the second integral in Eq. (4.44).

$$\int_0^x \frac{F_j^p(\xi)}{\bar{y}_j^p(\xi)}d\xi = \int_{t_j^-}^{t_j^p(x)} F^*(t)dt \leq \int_{t_j^-}^{\infty} |F(t)|\, dt. \tag{4.45}$$

Recall the equation describing the curve γ^+, that is the segment of the phase trajectory lying in the first quadrant, in the case of the instantaneous impact of intensity S $\quad (F(t) = S\delta(t))$. This curve satisfies the initial-value problem (see also Eq. (4.33)).

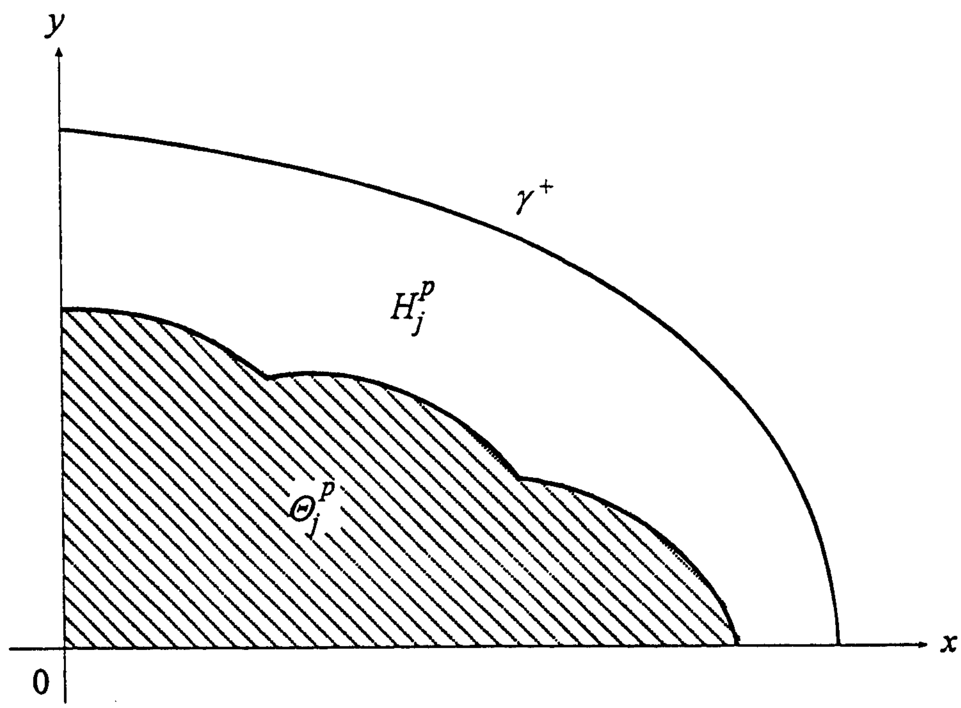


Figure 4-9. Region Θ_j^p.

$$\frac{dy}{dx} = -\frac{h + cy + \varphi(k,x)}{y}, \quad y(0) = S. \tag{4.46}$$

This initial-value problem can be represented in the integral form

$$y_0^+(x) = S - \int_0^x z[\xi, y_0^+(\xi)]d\xi. \tag{4.47}$$

Subtracting Eq. (4.44) from Eq. (4.47) we obtain

$$y_0^+(x) - \bar{y}_j^p(x) = S - y_j^- - \int_0^x \frac{F_j^p(\xi)}{\bar{y}_j^p(\xi)} d\xi - \int_0^x \eta[\xi, \bar{y}_j^p(\xi), y_0^+(\xi)]d\xi, \tag{4.48}$$

$$\eta[x, \bar{y}_j^p(x), y_0^+(x)] = z[x, y_0^+(x)] - z[x, \bar{y}_j^p(x)].$$

Now, we will prove that $y_0^+(x) - \bar{y}_j^p(x) \geq 0$ for $x \in [0, x_n^+]$. This implies that the representative point of the system cannot leave the set μ^* in the first quadrant. Note that $y_0^+(0) - \bar{y}_j^p(0) = S - y_j^- \geq 0$ according to Lemma 4.1.

Consider first the case $y_j^- = y(t_j^-; F) = S$. It follows from Eqs. (4.27) and (4.30) that

$$|\dot{x}(t;F)| = |y(t;F)| \leq \int_0^t |F(\tau)|d\tau. \tag{4.49}$$

Hence, in the case in question, $\int_0^{t_j^-} |F(\tau)|d\tau = S$ and $F(t) \equiv 0$ for $t > t_j^-$. This implies that for $t > t_j^-$, the system moves along the curve γ^+ satisfying Eq. (4.46) and, hence, its representative point does not leave the set μ^*.

Now, let $y_j^- < S$. Suppose that the inequality $y_0^+(x) - \bar{y}_j^p(x) \geq 0$ is violated at some point $x \in [0, x_n^+]$. Since functions $y_0^+(x)$ and $\bar{y}_j^p(x)$ are continuous, this assumption implies that there exists a point $x^* \in [0, x_n^+]$ such that $y_0^+(x^*) = \bar{y}_j^p(x^*)$, whereas $y_0^+(x) > \bar{y}_j^p(x)$ for $x \in [0, x^*)$. In this case we have

$$\eta[x, \bar{y}_j^p(x), y_0^+(x)] = [h + \varphi(k,x)] \frac{\bar{y}_j^p(x) - y_0^+(x)}{\bar{y}_j^p(x) y_0^+(x)} < 0, \qquad \text{for} \quad x \in [0, x^*). \tag{4.50}$$

Use the inequalities of Eqs. (4.45), (4.49), and (4.6) to obtain the estimate

$$S - y_j^- - \int_0^x \frac{F_j^p(\xi)}{\bar{y}_j^p(\xi)} d\xi \geq S - \int_0^{t_j^-} |F(\tau)|\, d\tau - \int_{t_j^-}^{\infty} |F(\tau)|\, d\tau \tag{4.51}$$

$$= S - \int_0^{\infty} |F(\tau)|\, d\tau \geq 0$$

Setting $x = x_*$ in Eq. (4.48) we find, taking into account Eqs. (4.50) and (4.51), that the right-hand side in Eq. (4.48) is positive, whereas the left-hand side is zero, according to our assumption. This contradiction proves that the inequality $y_0^+(x) - y_j^p(x) \geq 0$ does hold for all $x \in [0, x_n^+]$.

4.2.1.12 Principal Result. Prove now that the worst disturbance of the class of Eq. (4.6) for any isolator characteristic of the form of Eq. (4.22) is an instantaneous impact of intensity S, i.e., $F_1^0 = F_2^0 = \pm S\delta(t)$. Here, F_1^0 is the solution to Problem 4.3, and F_2^0 is the solution to Problem 4.4. It follows from Lemma 4.2 that the maximum displacement $J_1(u, F)$ of the body being isolated is achieved for the instantaneous impact of intensity S. It will be shown that the instantaneous impact of the intensity S also maximizes the peak acceleration $J_2(u, F)$ of the body.

It follows from the form of the characteristic given by Eq. (4.22), the assumed properties of the function $\varphi(k, x)$, and the structure of the set μ^* shown in Fig. 4.6 that

$$\max_{(x,y)\in\mu^*_{1,3}} |u(x, y; h, c, k)| = \max_{x\in[0,x_B]} \left|h + cy_0^+(x) + \varphi(k, x)\right| = \\ = \max_{x\in[-x_B,0]} \left|-h + cy_0^-(x) - \varphi(k, x)\right|, \tag{4.52}$$

$$\max_{(x,y)\in\mu^*_{2,4}} |u(x, y; h, c, k)| \leq \max\{h + cS, \varphi(k, x_B)\}. \tag{4.53}$$

Here, $\mu^*_{1,3}$ ($\mu^*_{2,4}$) is the portion of the set μ_* lying in the first and the third (the second and the fourth) quadrants of the phase plane x, y; x_B is the abscissa of the point B of the boundary of the set μ^* (see Fig. 4.6); $y_0^+(x)$ and $y_0^-(x)$ are the functions describing the curves γ^+ and γ^-, respectively. Obviously, $\mu^*_{1,3} \cup \mu^*_{2,4} = \mu^*$.

It follows from Eqs. (4.52) and (4.53) and the definition of the functions $y_0^+(x)$ and $y_0^-(x)$ that

$$\max_{(x,y)\in\mu^*} |u(x, y; h, c, k)| = \max_{x\in[0,x_B]} \left|h + cy_0^+(x) + \varphi(k, x)\right| = \\ = \max_{x\in[-x_B,0]} \left|-h + cy_0^-(x) - \varphi(k, x)\right| = J_2(u, S\delta(t)) = J_2(u, -S\delta(t)) \tag{4.54}$$

Since for any motion of the system of Eq. (4.31) the phase point (x, y) does not leave the set μ^*, we have

$$\max_{F\in\Phi} J_2(u, F) = J_2(u, S\delta(t)) = J_2(u_2, -S\delta(t)). \tag{4.55}$$

Thus, we have proved that solutions to both Problems 4.3 and 4.4 are given by the instantaneous impact of the maximum allowable intensity, i.e. $F_1^0 = \pm S\delta(t)$ and $F_2^0 = \pm S\delta(t)$, for any isolator whose characteristic has the form of Eq. (4.22).

4.2.1.13 An Example of an Isolator for which the Instantaneous Impact is not the Worst Disturbance. It can be shown that the result of the previous subsection is nontrivial. To do so, we Introduce an example of a system for which the instantaneous impact of maximum intensity is not The worst disturbance. Consider the system of Eq. (4.1) and choose as the isolator characteristic $u(x, \dot{x}, t) = -b\dot{x}|\dot{x}| - c\dot{x}$, and as the initial conditions $x^0 = 0$, and $\dot{x}^0 = 0$. The constants b and c

are positive. In this case, the isolator involves quadratic and linear dampers, but no stiffness element. Since the isolator contains a quadratic damper, it does not belong to the parametric family of Eq. (4.22). First, let $F(t) = S\delta(t)$. Without loss of generality we assume $S > 0$. In this case, the motion of the system is governed by the initial-value problem

$$\ddot{x} + b\dot{x}|\dot{x}| + c\dot{x} = 0, \quad x(0) = 0, \quad \dot{x}(0) = S \tag{4.56}$$

The motion of this system will continue until the velocity $\dot{x}$ vanishes and then will stop. During the motion, the coordinate x (the displacement of the body being isolated) monotonically increases, while the velocity monotonically decreases from the positive value S to zero. Since the velocity is nonnegative, the absolute value sign in Eq. (4.56) can be omitted. By considering x as a new independent variable, the system of Eq. (4.56) is reduced to the first-order system with respect to the velocity $\dot{x}$:

$$\frac{d\dot{x}}{dx} + b\dot{x} + c = 0, \quad \dot{x}(0) = S. \tag{4.57}$$

This initial-value problem governs the phase trajectory of the system of Eq. (4.56). The solution to the problem of Eq. (4.57) is given by

$$\dot{x} = (\frac{c}{b} + S)e^{-bx} - \frac{c}{b}. \tag{4.58}$$

Let us find the maximum value of the coordinate x which we will denote by D_+. Since the maximum x corresponds to $\dot{x} = 0$, we obtain D_+ by setting $\dot{x} = 0$ in Eq. (4.58) and solving the resulting equation for x. This yields

$$D_+ = \frac{1}{b} \ln \frac{c + Sb}{c}. \tag{4.59}$$

Consider now the motion of the system when subjected to the external disturbance

$$F(t) = \begin{cases} f_0, & \text{for} \quad 0 \le t \le \tau^* \\ 0, & \text{for} \quad t > \tau^* \end{cases}, \tag{4.60}$$

where constant $f_0 > 0$ and $\tau^* > 0$ are related by $f_0\tau^* = S$. On the time interval $[0, \tau^*]$, the motion is described by the initial-value problem

$$\ddot{x} + b\dot{x}^2 + c\dot{x} = f_0, \quad x(0) = 0, \quad \dot{x}(0) = 0. \tag{4.61}$$

Equation (4.61) is a separable equation for the velocity $\dot{x}$. By integrating this equation and taking into account the second initial condition in Eq. (4.61) we obtain the solution, which in an implicit form is given by

$$t = t(\dot{x}) = \frac{1}{(c^2 + 4bf_0)^{1/2}} \ln[\frac{(\dot{x} + v_+)v_-}{(\dot{x} + v_-)v_+}], \tag{4.62}$$

where

$$v_\pm = [c \pm (c^2 + 4bf_0)^{1/2}]/(2b).$$

Let us now find the coordinate x (the displacement of the body being isolated) at the time $t = \tau^* = S/f_0$

$$x(\tau^*) = \int_0^{\tau^*} \dot{x}(t)dt = \int_0^{\dot{x}(\tau^*)} \dot{x}\frac{dt(\dot{x})}{d\dot{x}}d\dot{x} \tag{4.63}$$

$$= \frac{1}{(c^2 + 4bf_0)^{1/2}}[v_- \ln \frac{\dot{x}(\tau^*) + v_-}{v_-} - v_+ \ln \frac{\dot{x}(\tau^*) + v_+}{v_+}].$$

The value of $\dot{x}(\tau^*)$ is found by solving Eq. (4.61) for $\dot{x}$ and setting $t = \tau^* = S/f_0$. It is given by

$$\dot{x}(\tau^*) = \frac{v_+ v_-(e^\lambda - 1)}{(v_- - v_+ e^\lambda)}, \quad \lambda = \frac{S(c^2 + 4bf_0)^{1/2}}{f_0}. \tag{4.64}$$

Let $f_0 \to 0$ in Eqs. (4.63) and (4.64), to find that

$$\lim_{f_0 \to 0} \dot{x}(\tau^*) = 0, \quad \lim_{f_0 \to 0} x(\tau^*) = \frac{S}{c}. \tag{4.65}$$

Denote $D_* = S/c$ and compile the ratio

$$\frac{D_+}{D_*} = \frac{\ln(1 + \alpha)}{\alpha} < 1, \quad \alpha = \frac{Sb}{c}, \tag{4.66}$$

where D_+ is defined in Eq. (4.59).The inequality in Eq. (4.66) shows that in this example the maximum displacement caused by the disturbance of Eq. (4.60) can exceed the maximum displacement corresponding to the instantaneous impact $F(t) = S\delta(t)$. Note that both of the disturbances belong to the class Φ defined by Eq. (4.6).

4.2.2 Limiting Isolation Capabilities.

Let us return to the problems of finding isolator characteristics providing the guaranteed minimum of the isolation performance criterion (Problems 4.1 and 4.2). Let the class Φ of possible external disturbances be defined by Eq. (4.6), and the class Y of admissible isolator characteristics by Eq. (4.22). As indicated in Section 4.2.1, these problems are then reduced to the problems of optimal protection against an instantaneous impact of the maximum allowable intensity, i.e. $F = S\delta(t)$. The isolator design problems for a prescribed impulse disturbance have been treated in depth in Chapter 2. According to the results of Section 2.3.5, all isolators of the class of Eq. (4.22) with $c = 0$, $\varphi(k, x) = k$, and $h + k = U$ are solutions to Problem 4.1, i.e.,

$$u_0 = hq + k\mathrm{sign}(x), \quad h + k = U, \tag{4.67}$$

where q is the dry-friction characteristic defined in Eq. (4.22). In this case, the maximum (over $F \in \Phi$) of the performance criteria $J_1(u, F)$ and $J_2(u, F)$ (see Eqs. (4.2) and (4.3)) are given by

$$\begin{aligned} &\max_{F\in\Phi} J_1(u_0, F) = J_1(u_0, S\delta(t)) = \tfrac{S^2}{2U}, \\ &\max_{F\in\Phi} J_2(u_0, F) = J_2(u_0, S\delta(t)) = U. \end{aligned} \tag{4.68}$$

The characteristic of Eq. (4.67) corresponds to the passive isolator consisting of a dry-friction damper and a bang-bang spring, with the stiffness and the damping factors being appropriately adjusted. In particular, for $h = 0$ and $k = U$ we have the undamped isolator with a bang-bang spring, whereas for $h = U$ and $k = 0$ we have the isolator with the dry-friction damper alone.

The particular values of the performance criteria indicated in Eq. (4.68) characterize the limiting capabilities of protection against the shock disturbance $F(t) = S\delta(t)$. See Chapter 2, Section 2.2. Hence, according to the results of Section 4.2.1, the isolators with the characteristic of Eq. (4.67) provide the limiting performance of protection against the disturbances of the class Φ defined in Eq. (4.6). It is worth noting here, that the characteristic of Eq. (4.67) is optimal not only for the class of characteristics of the form of Eq. (4.22) but for the class of all piecewise continuous functions of the phase variables of the system of Eq. (4.1). It follows from Eq. (4.68) that the optimal value of the criterion to be minimized (the maximum displacement of the body being isolated) monotonically decreases with the increase in the value of the constraint parameter U, while the constrained functional (the peak absolute acceleration) assumes its maximum allowable value. Hence, according to Theorem 1.1, Problem 4.2, which involves the minimization of the peak acceleration of the body to be isolated with the constraint imposed on its displacement, is the reciprocal of Problem 4.1. The optimal values of the performance criteria in Problem 4.2 are given by

$$\begin{aligned} &\max_{F\in\Phi} J_2(u^0, F) = J_2(u^0, S\delta(t)) = \tfrac{S^2}{2D}, \\ &\max_{F\in\Phi} J_1(u^0, F) = J_1(u^0, S\delta(t)) = D \end{aligned} \tag{4.69}$$

and the isolator characteristic that implements the optimum has the form

$$u^0 = hq + k\mathrm{sign}(x), \quad h + k = S^2/2D. \tag{4.70}$$

The isolators of the two-parameter family of Eq. (4.70) provide the limiting performance of isolation against the disturbances of the class of Eq.(4.6), in the case where the peak acceleration of the body being isolated is the optimization criterion.

Recall that among the isolators implementing the limiting capabilities of protection against the instantaneous impact of known intensity ($F(t) = S\delta(t)$), there is an isolator with a linear spring and a quadratic damper (see Chapter 2, Section 2.3.5). It is interesting to investigate whether this

isolator also provides the best possible protection against the class of integrally constrained disturbances of Eq. (4.6). The answer is no.

Consider, for example, Problem 4.2 for the case where the isolator consists of a linear spring and a quadratic damper and at the initial instant the body being isolated rests at the state $x(0) = 0, \quad \dot{x}(0) = 0$. In this case, the system of Eq. (4.1) has the form

$$\ddot{x} + c\dot{x}^2 \text{sign}(\dot{x}) + kx = F(t), \quad x(0) = 0, \quad \dot{x}(0) = 0, \tag{4.71}$$

where $c \geq 0$ and $k \geq 0$ are the damping and stiffness coefficients to be found as a result of solving the optimization problem.

Without loss of generality, assume $S = 1$ in Eq. (4.6) and $D = 1$ in the statement of Problem 4.2. To achieve this, introduce dimensionless (primed) variables and parameters

$$x' = \frac{x}{D}, \quad t' = \frac{S}{D}t, \quad c' = cD, \quad k' = k\frac{D^2}{S^2}, \quad F(t) = \frac{S^2}{D}F'(t'). \tag{4.72}$$

In the following, we use the dimensionless variables without primes.

It was shown in Chapter 2 (Section 2.3.5) that the isolator with a quadratic damper and a linear spring provides the best possible protection against the instantaneous impact of unit intensity ($F(t) = \delta(t)$) if the damping and the stiffness coefficients are $c = 0.5$ and $k = 0.5$. This isolator would provide the optimal protection against the disturbances of the class of Eq. (4.6) if the instantaneous impact $F(t) = \delta(t)$ were the worst (in the sense of Problems 4.3 and 4.4) disturbance. However, this is not the case.

Let us consider the disturbance defined in Eq. (4.60) where f_0 and τ^* are related by $f_0\,\tau^* = 1$ and calculate (numerically) the values of the criteria J_1 and J_2 (Eqs. (4.2) and (4.3)) for the system of Eq. (4.71) with $c = 0.5$ and $k = 0.5$. The results are presented in Table 4.1 for some values of τ^*.

Table 4-1. The peak displacement and the peak acceleration versus the disturbance duration for the system with a linear spring and a quadratic damper

τ^*	J_1	J_2
0.20	1.01	0.510
0.50	1.02	0.527
1.00	1.04	0.539
1.80	1.05	0.535

On the other hand, as shown in Section 2.3.5, if $F(t) = \delta(t), c = 0.5$, and $k = 0.5$ in the system of Eq. (4.71), then $J_1 = 1$ and $J_2 = 0.5$. By comparing these values with those given in the table, we see that there are disturbances belonging to the class of Eq. (4.6) that give rise to greater values for both the functionals J_1 and J_2 than the instantaneous impact generates. Thus, the isolator with a linear spring and a quadratic damper providing the optimal protection against the impact $F(t) = \delta(t)$ is not an optimal isolator for the class of integrally constrained disturbances.

This conclusion applies for Problem 4.2. It is also valid for Problem 4.1. This follows from the reciprocity (in the sense of Theorem 1.1) of the problems.

It was shown in Section 2.3.5 that among the passive isolators with power law characteristics only four kinds of isolators provide the best possible protection against the instantaneous impact. They are

1) the isolator with the linear spring and the quadratic damper;

2) the isolator with the dry-friction damper alone;

3) the isolator with the bang-bang spring alone;

4) the isolator with the dry-friction damper and the bang-bang spring.

Of these isolators, isolators (2) to (4) also provide the best possible protection against the class of Eq. (4.6) of integrally constrained disturbances. Isolator (1) does not possess this property.

4.2.3 Parametric Optimization.

Consider a special case of Problems 4.1 and 4.2 where the set Y of admissible isolator characteristics is a parametric family of functions depending on x and $\dot{x}$. In this case, the minimization in Eqs. (4.4) and (4.5) is performed with respect to a finite number of variables (the parameters of the family). However, in the general case, the problem is not reduced to the constrained minimization of a function of a finite number of variables, because for each candidate $u \in Y$ we have to determine the maxima of the performance indices J_1 and J_2 with respect to $F \in \Phi$ (Eqs. (4.4) and (4.5)). Since Φ is not a parametric family, this can involve a rather complicated variational problem (see also Section 1.3 of Chapter 1). The computational techniques of Chapter 8 provide versatile alternatives for solving this class of problems.

The situation is significantly simplified if it is possible to identify the disturbance F which is independent of u and maximizes both functionals $J_1(u, F)$ and $J_2(u, F)$ for all $u \in Y$. In this case the problem is reduced to the parametric optimization of the isolator characteristics.

In Section 4.2.1, it was shown that for all characteristics belonging to the class of Eq. (4.22), the worst disturbance of the class of Eq. (4.6) is the instantaneous impact $F = S\delta(t)$. Hence, if the parametric family Y is a subset of the class of Eq. (4.22), Problems 4.1. and 4.2 for the optimization of isolators for the class of disturbances are reduced to finding optimal parameters of the isolator to ensure the best protection from the instantaneous impact of given intensity. In other words, in this case, the problem is reduced to the constrained minimization of a function of a finite number of variables.

Consider the example of linear spring-and-damper isolators with the characteristics

$$u(x, \dot{x}, c, k) = c\dot{x} + kx, \quad c \geq 0, \quad k \geq 0. \tag{4.73}$$

It is evident that the parametric family of characteristics of Eq. (4.73) is a subset of the class of characteristics of Eq. (4.22) where $h = 0$ and $\varphi(k, x) = k|x|$. Hence, the optimal parameters c and

k of the isolator (4.73) are those providing the optimal protection of the object being isolated against the impact $F = S\delta(t)$ and we can make use of the results of Section 2.3.4.

The optimal parameters minimizing the peak acceleration under the constrained displacement of the object are given by Eq. (2.131). In these equations, set $S = 1$, $r = 1$, replace β by S, and substitute for $(c^0)'$ and $(k^0)'$ the appropriate values from Table 2.1. The corresponding optimal values of the performance criteria are given by Eq. (2.132). Thus, the optimal stiffness and damping factors of the linear isolator solving Problem 4.2 (for Y defined by Eq. (4.73)) are

$$c^0 \approx 0.485\frac{S}{D}, \quad k^0 \approx 0.361\frac{S^2}{D^2}, \tag{4.74}$$

and the optimal values of the performance criteria are expressed as

$$\max_{F\in\Phi} J_2(u^0, F) = J_2(u^0, S\delta(t)) = 0.521\frac{S^2}{D}, \tag{4.75}$$

$$\max_{F\in\Phi} J_1(u^0, F) = J_1(u^0, S\delta(t)) = D,$$

where $u^0 = c^0\dot{x} + k^0 x$.

Similarly, on the basis of Eq. (2.134), we obtain the optimal parameters of the linear isolator of Eq. (4.73) solving Problem 4.1

$$c_0 \approx 0.931\frac{U}{S}, \quad k_0 \approx 1.330\frac{U^2}{S^2} \tag{4.76}$$

and the corresponding optimal values of the performance criteria

$$\max_{F\in\Phi} J_1(u_0, F) = J_1(u_0, S\delta(t)) = 0.521\frac{S^2}{U} \tag{4.77}$$

$$\max_{F\in\Phi} J_2(u_0, F) = J_2(u_0, S\delta(t)) = U,$$

where $u_0 = c_0\dot{x} + k_0 x$.

Note that Problems 4.1 and 4.2 for Φ defined by Eq. (4.6) and Y defined by Eq. (4.73) are reciprocal to each other (in the sense of Theorem 1.1), and Eqs. (4.76) and (4.77) can be obtained from Eqs. (4.74) and (4.75) by applying Theorem 1.1.

The comparison of the values of Eqs. (4.75) and (4.77) of the performance criteria with the corresponding values in Eqs. (4.68) and (4.69) characterizing the limiting isolation capabilities shows that the optimal linear isolator provides rather effective protection, inferior to the limiting capabilities of isolation by only 4%.

4.3 OPTIMAL PROTECTION AGAINST SUCCESSIVE IMPACTS.

4.3.1 Problem Formulation.

In Chapter 2, we studied in some detail the problem of optimal protection of an object mounted on a rectilinearly moving base against a single instantaneous impact of known intensity. In actual operating conditions, a system with isolators can undergo several impacts separated by time intervals. As a rule, the directions and the intensities (absolute values of impact impulses) of the impacts, as well as the time intervals between them, are not known precisely, and one can give only some estimates for these characteristics of the applied disturbance.

Consider again the object to be isolated mounted on a rectilinearly moving base. The object can move relative to the base along the line of the latter's motion. The base is subjected to a succession of instantaneous impacts separated by time intervals. The body being isolated is assumed to rest (relative to the base) at the initial instant $t = 0$ at the position corresponding to $x = 0$. In this case, the relative motion of the object being isolated is governed by the initial value problem of Eq. (4.1), with $t_0 = 0, \quad x^0 = 0, \quad \dot{x}^0 = 0, \quad F(t) = \sum_{i=1}^{N} a_i\delta(t - T_i)$. That is,

$$\begin{gathered} \ddot{x} + u(x, \dot{x}, t) = \textstyle\sum_{i=1}^{N} a_i\delta(t - T_i), \\ x(0) = 0, \quad \dot{x}(0) = 0, \end{gathered} \tag{4.78}$$

where a_i is the increment of the relative velocity of the body resulting from the ith impact, T_i is the instant of the ith impact, and N is the number of impacts.

4.3.1.1 Problem 4.5. Find a piecewise continuous isolator characteristic $u(x, \dot{x}, t)$ which satisfies the constraint

$$|u(x, \dot{x}, t)| \leq U \tag{4.79}$$

and minimizes the functional

$$J(u) = \max_{T_i} \max_{a_i} \max_{t} |x(t)| \tag{4.80}$$

under the constraints

$$\begin{gathered} |a_i| \leq A, \quad T_1 = 0, \quad T_{i+1} - T_i \geq T_0, \\ (i = 1, 2, ..., N), \end{gathered} \tag{4.81}$$

where, U, A, and T_0 are specified positive numbers and $x(t)$ is the solution to the initial-value problem of Eq. (4.78). The constraints of Eq. (4.81) ensure that (1) the absolute value of the increment of the relative velocity of the body for each impact does not exceed A, (2) the first impact occurs at the initial instant $t = 0$ (this condition does not lead to a loss of generality), and (3) the time between two successive impacts is not less than T_0.

Problem 4.5 is a particular case of Problem 4.1. For this case, Y is the set of piecewise continuous functions of x, $\dot{x}$, and t, and Φ is the set of functions $F = \sum_{i=1}^{N} a_i\delta(t - T_i)$ satisfying constraints of Eq. (4.81).

Note that Problem 4.5 involves constructing the optimal isolator characteristic in a feedback form. There have been numerous efforts to find the optimal feedback characteristics of shock isolators for the case of complete information about the external disturbance. For example, in Troitskii (1967, 1976), the optimal characteristic of a passive isolator is determined which provides the minimum absolute value of the displacement of the body to be isolated under the constraint on its absolute acceleration, when the external disturbance is a single impact ($F(t) = \beta\delta(t)$). This characteristic has a form substantially different from the power law functions defined by Eqs. (2.30) and (2.112) and does not coincide with any of the optimal characteristics obtained in Chapter 2 for the same external disturbance.

In Bolychevtsev (1971) the feedback characteristic has been synthesized which ensures the best protection against periodically repeated impacts that have equal intensities and act in the same direction ($F(t) = a\sum_{n=0}^{\infty}\delta(t-nT)$), and in Bolychevtsev (1973) the optimal isolator has been developed for periodically repeated impacts having equal intensities and alternating directions ($F(t) = a\sum_{n=0}^{\infty}(-1)^n\delta(t-nT)$). Unlike the problems considered by Bolychevtsev, in Problem 4.5 neither the intensities and directions of individual impacts nor the time intervals between them are fixed.

4.3.2 Optimal Feedback Isolator for the Case of Two Impacts.

In what follows, we present the solution to Problem 4.5 for the case of two impacts ($N = 2$). Without loss of generality we set $U = 1$ and $A = 1$. This corresponds to the use of the dimensionless (primed) variables

$$\begin{array}{lll} x' = \frac{U}{A^2}x, & t' = \frac{U}{A}t, & T_i' = \frac{U}{A}T_i, \\ T_0' = \frac{U}{A}T_0, & a_i' = \frac{a_i}{A}, & u' = \frac{u}{U}. \end{array} \tag{4.82}$$

For convenience, we will use the dimensionless variables without primes.

We seek the optimal isolator characteristic in the form

$$u(x,\dot{x}) = \operatorname{sign}\{f(x,\dot{x})\}, \tag{4.83}$$

where $f(x,\dot{x})$ is a function of the phase coordinates to be determined. Then the problem reduces to constructing the switching curve $f(x,\dot{x}) = 0$ for the optimal characteristic. We will solve this problem by investigating the phase trajectories of the system in question.

4.3.2.1 Phase Trajectories. It follows from the equation of motion of Eq. (4.78) that during the time intervals between impacts, the representative (phase) point of the system moves along a parabola of the family

$$x = -\frac{\dot{x}^2}{2} + C, \tag{4.84}$$

if $u = 1$ or along a parabola of the family

$$x = \frac{\dot{x}^2}{2} + C, \tag{4.85}$$

if $u = -1$. These trajectories are shown in Fig. 4.10. The direction of the motion is indicated by the arrows. Equations (4.78) also imply that the time t_{12} of motion between two points along one of the parabolas of Eq. (4.84) or (4.85) is equal to the absolute value of the difference of the velocities corresponding to these points, i.e.,

$$t_{12} = |\dot{x}_1 - \dot{x}_2|\,. \tag{4.86}$$

4.3.2.2 Solution of Problem 4.5 for $T_0 \geq 1 + \sqrt{2}$.It is shown in Troitskii (1967, 1976) that for a single impact ($N = 1$ in the formulation of Problem 4.5), the isolator $u(x, \dot{x})$ =sign$(x - l(\dot{x}))$ with the switching curve

$$x = l(\dot{x}) = \begin{cases} -\frac{\dot{x}^2}{2}, & \text{if } \dot{x} \geq 0 \\ \frac{\dot{x}^2}{2}, & \text{if } \dot{x} < 0 \end{cases} \tag{4.87}$$

is optimal. The curve of Eq. (4.87) coincides with the switching curve of the optimal control force driving a single-degree-of-freedom particle of unit mass to the origin of the phase plane $\{x, \dot{x}\}$ in minimal time, provided the absolute value of the control force is bounded by unity (Pontryagin, et al, 1962). The corresponding maximum (over time) of the absolute value of the displacement is equal to $a_1^2/2$ and the representative point of the system comes to the origin in the time $t_* = |a_1|\,(1 + \sqrt{2})$. Time t_* is the minimum possible time of arrival at the origin. This indicates that in case the system is subjected to only one impact, the worst disturbance, leading to the maximum of the peak displacement, corresponds to $|a_1| = 1$. If $T_0 \geq 1 + \sqrt{2}$, then the isolator with the switching curve of Eq. (4.87) is optimal for an arbitrary number of impacts. In this case, the optimal value of the criterion of Eq. (4.80) is equal to 1/2.

4.3.2.3 Solution of Problem 4.5 for $0 < T_0 < 1 + \sqrt{2}$. Sufficient Conditions of Optimality. Let the coordinates of the phase point at the instant of the second impact be x and $\dot{x}$. The worst impact corresponds to $|a_1| = 1$, since in this case the phase point jumps to the parabola of the family of Eq. (4.84) or (4.85) with maximum $|C|$ which corresponds to the maximum absolute value of the displacement.

As a result of the impact with $a_2 = 1$, the phase point can jump to the trajectory of Eq. (4.84) with

$$C = C_1 = x + \frac{(\dot{x} + 1)^2}{2} \tag{4.88}$$

or to the trajectory of Eq. (4.85) with

$$C = C_2 = x - \frac{(\dot{x} + 1)^2}{2}. \tag{4.89}$$

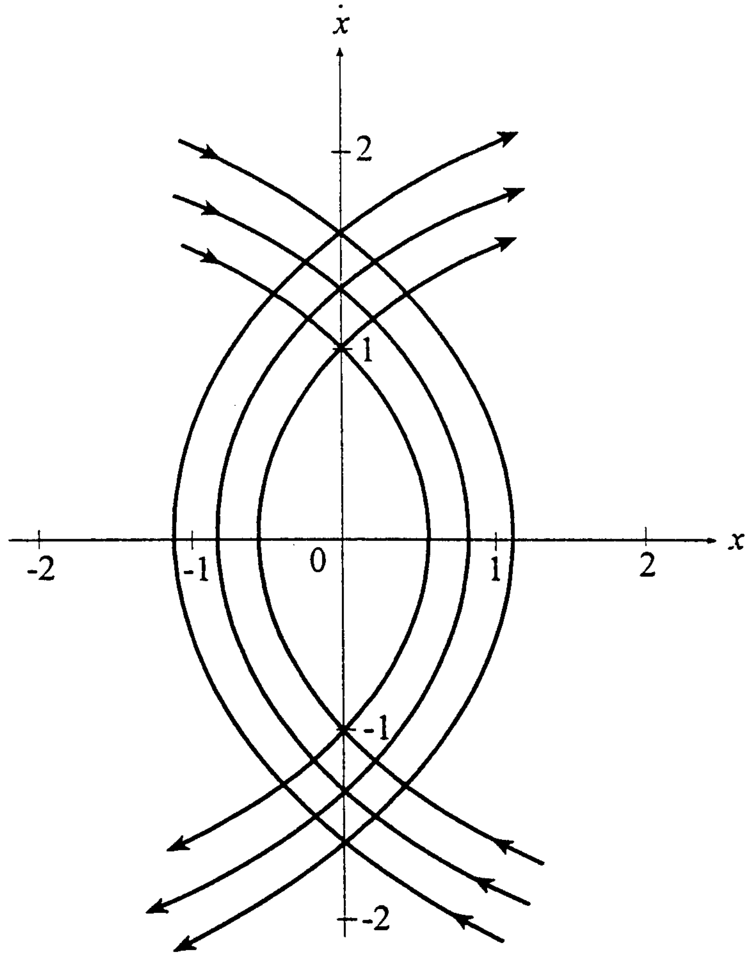


Figure 4-10. Phase trajectories of the system $\ddot{x}+u=\sum_{i=1}^{N} a_i\,\delta(t-T_i)$ during the time intervals between impacts for $u=\pm 1$.

As a result of the impact with $a_2 = -1$, the phase point can jump to the trajectory of Eq. (4.84) with

$$C = C_3 = x + \frac{(\dot{x} - 1)^2}{2} \tag{4.90}$$

or to the trajectory of Eq. (4.85) with

$$C = C_4 = x - \frac{(\dot{x} - 1)^2}{2}. \tag{4.91}$$

Introduce the notation

$$\eta(x, \dot{x}) = \{\min_{u} \max_{a_2} \max_{t} |x(t)| \quad |x(T_2) = x, \quad \dot{x}(T_2) = \dot{x}\}, \tag{4.92}$$

where T_2 is the instant of the second impact. The expression in the curly brackets denotes the minmax of the function $|x(t)|$ calculated under the condition that at the instant of the second impact the phase coordinates of the system are equal to x and $\dot{x}$. Denote by $\Omega(T, a_1)$ the set of the phase points to which the system can come in a time not exceeding T after the first impact, provided the body being isolated has acquired the velocity $\dot{x} = a_1$ as a result of this impact.

By examining various directions of the second impact and analyzing the corresponding phase trajectories one can prove that

$$\eta(x, \dot{x}) = \max\{|C_1|, |C_4|\}, \quad \text{if} \quad (x, \dot{x}) \in \omega, \tag{4.93}$$

$$\min_{(x,\dot{x}) \in \Omega(T,a_1) \cap \omega} \eta(x, \dot{x}) = \min_{(x,\dot{x}) \in \Omega(T,a_1)} \eta(x, \dot{x}), \tag{4.94}$$

$$\min_{(x,\dot{x}) \in \Omega(T,1) \cup \Omega(T,-1)} \eta(x, \dot{x}) > \min_{(x,\dot{x}) \in \cup_{|\xi|<1} \Omega(T,\xi)} \eta(x, \dot{x}), \tag{4.95}$$

where,

$$\begin{gathered} \omega = \{x > 0, \dot{x} > 0\} \cup \{x < 0, \dot{x} < 0\} \cup \{-1 < x < 0, 0 < \dot{x} < 1\} \cup \\ \cup \{0 < x < 1, -1 < \dot{x} < 0\} \end{gathered} \tag{4.96}$$

The inequality of Eq. (4.95) indicates that the least favorable first impact corresponds to $|a_1| = 1$. Without loss of generality we will assume $a_1 = 1$. This can always be achieved by a proper choice of the x-axis direction. It follows from Eqs. (4.88), (4.91), and (4.93) that $\eta(x, \dot{x}) \geq 1/2$, the equality holding if and only if $x = \dot{x} = 0$. This implies that the optimal value of the criterion J in Eq. (4.80) is equal to the corresponding value of $\eta(x, \dot{x})$ and that the optimal value of the performance criterion in the case of two impacts is not less than in the case of a single impact.

On the basis of the above arguments, we can formulate the sufficient optimality conditions for the isolator.

Proposition 4.1. The isolator is optimal if it satisfies the following conditions:

1. At the instant T_0 (the nearest possible instant of the second impact), the system is at point $(x^*, \dot{x}^*)$ of the region $\Omega(T_0, 1)$ corresponding to the minimum value of $\eta(x, \dot{x})$.

2. If the second impact did not occur at the instant T_0, then $\eta(x(t), \dot{x}(t)) \leq \eta(x^*, \dot{x}^*)$ for $t > T_0$ until the second impact occurs.

3. The value of $|x(t)|$ does not exceed $\eta(x^*, \dot{x}^*)$ at any time instant.

By virtue of Eq. (4.94) we can change $\Omega(T_0, 1)$ for $\Omega(T_0, 1) \cap \omega$ in condition 1. Accordingly, we will consider only the isolators satisfying the inclusion $(x(T_0), \dot{x}(T_0)) \in \Omega(T_0, 1) \cap \omega$.

Switching Curve of the Optimal Isolator. Consider the isolator with the characteristic $u_0(x, \dot{x})$ =sign$(x - \psi(\dot{x}))$ whose switching curve is given by

$$x = \psi(\dot{x}) = \begin{cases} \frac{\dot{x}^2}{2} + \frac{1}{2}[1 - T_0^2(3 - 2\sqrt{2})], & \text{if} \quad \dot{x} < -x_0 \\ -x, & \text{if} \quad -x_0 \leq \dot{x} \leq x_0 \\ -\frac{\dot{x}^2}{2} - \frac{1}{2}[1 - T_0^2(3 - 2\sqrt{2})], & \text{if} \quad \dot{x} > x_0 \end{cases} , \tag{4.97}$$

where

$$x_0 = 1 - T_0(\sqrt{2} - 1). \tag{4.98}$$

The curve of Eq. (4.97) is shown in Fig. 4.11. It is symmetric with respect to the coordinate origin and consists of the segment of the bisector of the second and fourth coordinate angles and two parabolas. If $T_0 \leq \sqrt{2}$ ($\sqrt{2}$ is the time of motion from the point $(0, 1)$ along the parabola $x = -\dot{x}^2/2 + 1/2$ until the phase trajectory intersects the line $x = -\dot{x}$), then the isolator with the characteristic $u_0(x, \dot{x})$ brings the system for the time T_0 from the position $(0, 1)$ to the point with the coordinates $x_* = -(1 - T_0)^2/2 + 1/2$, $\quad \dot{x}_* = 1 - T_0$. In this case, the representative point moves along the parabola $x = -\dot{x}^2/2 + 1/2$. If $T_0 > \sqrt{2}$, then the isolator with the characteristic $u_0(x, \dot{x})$ brings the system for the time T_0 from the point $(0, 1)$ to the point $(x_0, -x_0)$. In this case, the representative point first moves along the parabola $x = -\dot{x}^2/2 + 1/2$ until it intersects the parabolic part of the curve of Eq. (4.97) and then goes along the parabolic part to the position $(x_0, -x_0)$.

For further analysis, let us recall some properties of the time-optimal control, driving the system $\ddot{x} + u = 0$ from one point on the phase plane to another under the constraint $|u| \leq 1$. It is well known, for example, from Pontryagin, et al, (1962), that the optimal control is equal to either $+1$ or -1, has at most one switching point, and the switching curve consists of two parabolic parts, trajectories of the families of Eqs. (4.84) and (4.85) leading to the desired point P (Fig. 4.12). The phase trajectory of the optimal motion consists of at most two parts of parabolas of the families of Eqs. (4.84) and (4.85), the part leading to the desired point lies on the switching curve. This, in particular, implies that T_0 is the shortest time for driving the system from the position $(0, 1)$ to the position $(x_0, -x_0)$ if $T_0 > \sqrt{2}$.

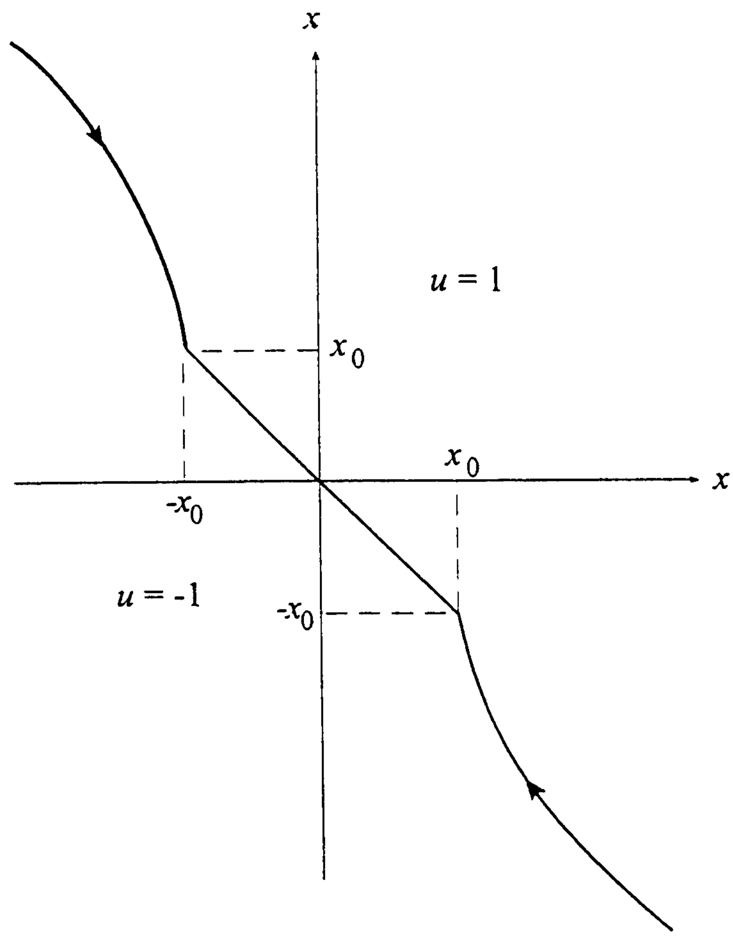


Figure 4-11. Switching curve of the optimal control.

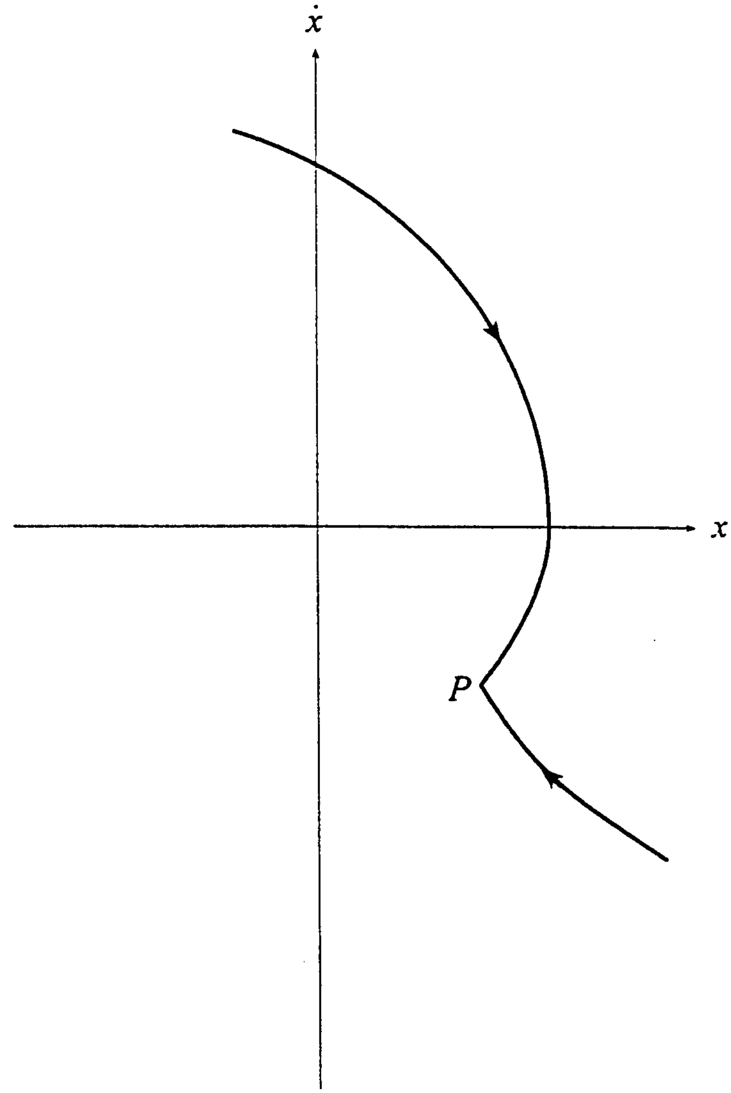


Figure 4-12. Optimal phase trajectory for the system $\ddot{x}+u=0$ constrained by $|u| \leqslant 1$.

Let us show that the isolator with the characteristic $u_0(x,\dot{x})$ is the optimal isolator. From Eqs. (4.88), (4.91), and (4.93), it follows that

$$\eta(x,\dot{x}) = \begin{cases} |C_4|, & \text{if} \quad x < -\dot{x} \\ |C_1|, & \text{if} \quad x > -\dot{x} \\ |C_1| = |C_4| = \frac{\dot{x}^2}{2} + \frac{1}{2}, & \text{if} \quad x = -\dot{x} \end{cases} \tag{4.99}$$

$$\frac{\partial\eta(x,\dot{x})}{\partial x} = \begin{cases} -1, & \text{if} \quad x < -\dot{x} \\ 1, & \text{if} \quad x > -\dot{x} \\ x, & \text{if} \quad x = -\dot{x} \end{cases} .$$

If $T_0 \le \sqrt{2}$, then

$$\eta(x_*,\dot{x}_*) = \min_{(x,\dot{x})\in\Omega(T_0,1)} \eta(x,\dot{x}), \tag{4.100}$$

where

$$x_* = -(1-T_0)^2/2 + 1/2, \quad \dot{x}_* = 1 - T_0. \tag{4.101}$$

Indeed, it follows from Eqs. (4.101), (4.99), and (4.88) that $\eta(x_*,\dot{x}_*) = x_* + (\dot{x}_* + 1)^2/2$. The aforementioned properties of the time-optimal motion and the relation of Eq. (4.86) ensure that the phase trajectory cannot intersect the line $x = -\dot{x}$ for the time less than $\sqrt{2}$. Hence, if $T_0 \le \sqrt{2}$, then the phase points $(x,\dot{x})$ satisfying the relation $\eta(x,\dot{x}) < \eta(x_*,\dot{x}_*)$ lie in the region defined by the inequalities

$$x + \frac{(\dot{x}+1)^2}{2} < x_* + \frac{(\dot{x}_*+1)^2}{2}, \quad \dot{x} \ge -x \tag{4.102}$$

or, which is equivalent, by the inequalities

$$x + \frac{(\dot{x}+1)^2}{2} - \dot{x}_* - 1 < 0, \quad \dot{x} \ge -x. \tag{4.103}$$

This region is shown shaded in Fig. 4.13. It follows from the properties of the time-optimal motion and from Eq. (4.86) that the phase point cannot enter the shaded region for the time not exceeding T_0. This completes the proof of the relation of Eq. (4.100). Similar arguments, although somewhat more complicated, prove that if $T_0 > \sqrt{2}$, then

$$\eta(x_0,-x_0) = \min_{(x,\dot{x})\in\Omega(T_0,1)} \eta(x,\dot{x}), \tag{4.104}$$

where x_0 is defined by Eq. (4.98). Hence, the isolator with the switching curve of Eq. (4.97) satisfies condition 1 of Proposition 4.1, with $x^* = x_*$ and $\dot{x}^* = x_*$, for $T_0 \le \sqrt{2}$, and $x^* = x_0$ and $\dot{x}^* = -x_0$, for $T_0 > \sqrt{2}$. It also satisfies conditions 2 and 3. This follows from the shape of the

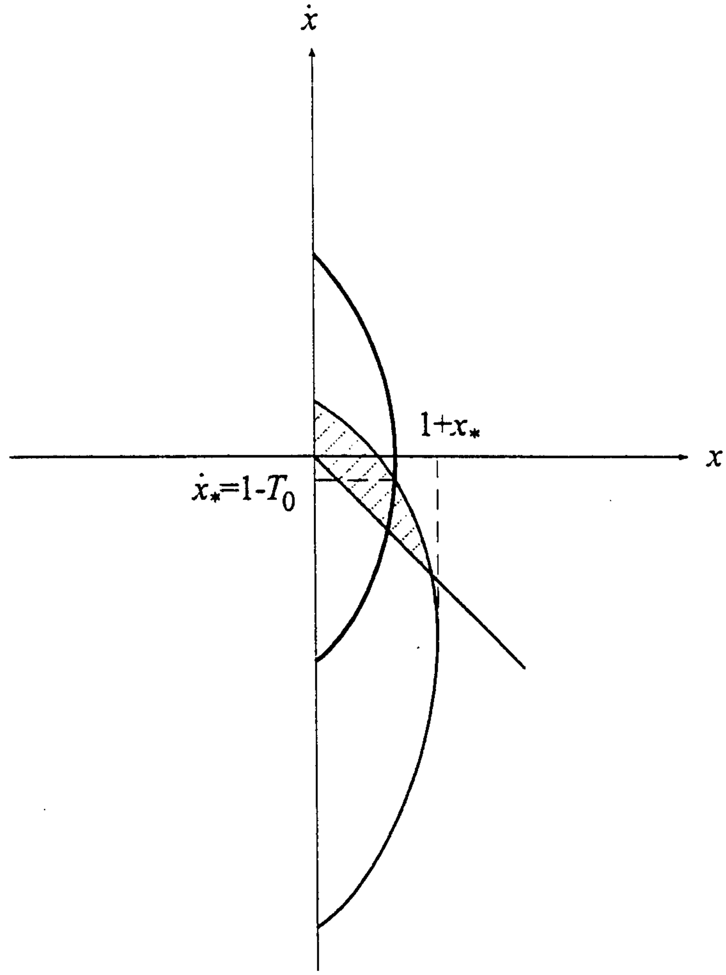


Figure 4-13. The region $\eta\,(x, \dot{x}) < \eta\,(x_*, \dot{x}_*)$ for $T_0 \leq \sqrt{2}$.

phase trajectories corresponding to this isolator and from the relation of Eq. (4.99). Hence, the isolator with the characteristic $u_0(x, \dot{x})$ =sign$(x - \psi(\dot{x}))$ is optimal.

The value of the performance criterion of Eq. (4.80) corresponding to the optimal isolator is given by

$$J(u_0) = \begin{cases} 2 - T_0, & \text{if } T_0 < \sqrt{2} \\ 1 + \frac{T_0^2}{6+4\sqrt{2}} - \frac{T_0}{1+\sqrt{2}}, & \text{if } \sqrt{2} \le T_0 \le 1 + \sqrt{2} \\ 1/2 & \text{if } T_0 > 1 + \sqrt{2} \end{cases} . \tag{4.105}$$

Note that $x_0 \to 0$ as $T_0 \to 1 + \sqrt{2}$ and the switching curve of Eq. (4.97) tends to the switching curve of Eq. (4.87) corresponding to the control driving a particle to the coordinate origin in minimal time. It is evident from Eq. (4.105) that $J(u_0) \to 1/2$ as $T_0 \to 1 + \sqrt{2}$.

Note also the presence of the sliding mode in the operation of the optimal isolator. The sliding mode occurs when the phase point moves along the bisector of the second and fourth coordinate angles.

4.4 SUMMARY OF THE BASIC RESULTS.

The single-degree-of-freedom system with the isolator consisting of a linear or nonlinear spring, a viscous linear damper, and a Coulomb damper was considered. The system is subject to an unknown external disturbance which is assumed to belong to a prescribed class. This class of disturbances is defined as functions of time for which the integral of the absolute value does not exceed a specified number. It is established that the worst disturbance of this class is the instantaneous impact with the maximum allowable intensity. It is important that the instantaneous impact with the maximum allowable intensity provides the maximum value for both the peak displacement of the body to be isolated and the peak force transmitted to the body for any stiffness and damping coefficients of the isolator. Note that this property is proven only for the isolator consisting of a linear or nonlinear spring, a viscous linear damper, and a Coulomb damper and is not valid in the general case. For example, it is shown that for the system consisting of a linear spring and a quadratic law damper, the instantaneous impact with the maximum allowable intensity is not the worst disturbance.

The isolator consisting of the bang-bang spring and the Coulomb damper with the corresponding stiffness and damping coefficients implements the limiting isolation capabilities for the class of disturbances subject to the integral constraint.

The problem of the optimal isolation of a single-degree-of-freedom system for the case where the external disturbance is a series of instantaneous impacts was considered. The intensities and directions of these impacts, as well as the time intervals between them are not completely prescribed. It is known only that the intensity of each of the impacts does not exceed a specified value and the time interval between two successive impacts is bounded from below by a prescribed number. The force transmitted to the body to be isolated is subject to the constraint. The bang-bang feedback control of the isolator providing the minimum for the peak displacement of the body is constructed for the case of a series of two impacts.

SECTION 5

OPTIMIZATION OF SHOCK ISOLATORS FOR AN OBJECT WITH INCOMPLETELY PRESCRIBED MASS

5.1 STATEMENT OF THE PROBLEM.

Consider a single-degree-of-freedom system subjected to an instantaneous impact of intensity β (Fig. 5.1). This system is governed by the initial value problem (see also Eqs. (2.4) and (2.13))

$$m\ddot{x} + f(x, \dot{x}) = 0, \quad x(0) = 0, \quad \dot{x}(0) = \beta, \tag{5.1}$$

where $f(x, \dot{x})$ is the passive isolator characteristic (the force applied to the body by the isolator) and m is the mass of the body. We assume that $f(x, \dot{x})$ is a function of a certain class Y, which will be specified in what follows. The mass m is the unknown parameter that can take values in a prescribed interval $m_1 \leq m \leq m_2$.

We restrict our consideration to passive isolators having power law characteristics of the form

$$f(x, \dot{x}) = c|\dot{x}|^r \text{sign}(\dot{x}) + k|x|^n \text{sign}(x), \quad r \geq 2, \quad n \geq 1. \tag{5.2}$$

Thus, the class Y is a four-parameter (c, k, r and n) family of functions and the initial value problem of Eq. (5.1) becomes

$$m\ddot{x} + c|\dot{x}|^r \text{sign}(\dot{x}) + k|x|^n \text{sign}(x) = 0, \quad r \geq 2, \quad n \geq 1, \tag{5.3}$$

$$x(0) = 0, \quad \dot{x}(0) = \beta, \quad m \in [m_1, m_2].$$

Note that the isolator characteristics specified by Eq. (5.2) satisfy Assumptions 1 to 5, enumerated in Section 2.3.2, and the inequality of Eq. (2.77). For such characteristics, all properties of the performance indices established in Section 2.3.2 are valid.

Let us fix the parameters r and n and consider the optimization with respect to the damping coefficient c and the stiffness coefficient k. The performance criteria are specified as

$$I_1(c, k, m) = \max_{t \in [0,\infty)} |x(t; c, k, m)|, \tag{5.4}$$

$$I_2(c, k, m) = \max_{t \in [0,\infty)} |\ddot{x}(t; c, k, m)| = \tag{5.5}$$

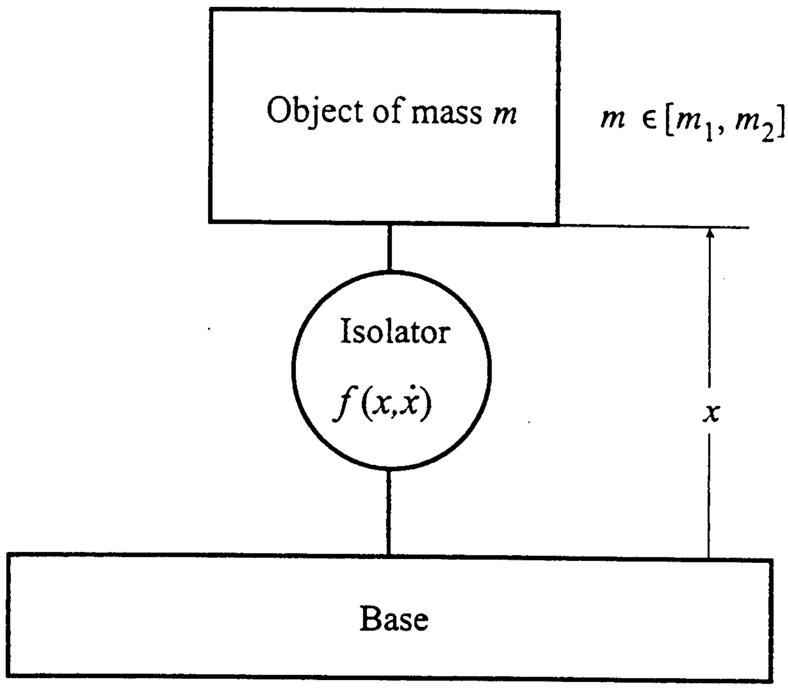


Figure 5-1. Single-degree-of-freedom model for an object connected to a moving base with a passive isolator.

$$\frac{1}{m}\max_{t\in[0,\infty)}|c|\dot{x}(t;c,k,m)|^r \mathrm{sign}(\dot{x}(t;c,k,m))+k|x(t;c,k,m)|^n \mathrm{sign}(x(t;c,k,m))|,$$

where $x(t;c,k,m)$ denotes the solution of the initial value problem of Eq. (5.3). In what follows, we will often omit the parameters c, k,and m in the list of the arguments of the function x so that $x(t;c,k,m)$ will be written as $x(t)$. Criterion I_1 defines the maximum value of the displacement of the body being isolated, while criterion I_2 the maximum value of its absolute acceleration.

5.1.0.1 Problem 5.1. Find parameters $c_0 \geq 0$ and $k_0 \geq 0$ such that

$$\max_{m\in[m_1,m_2]} I_1(c_0,k_0,m) = \min_{c\geq 0, k\geq 0}\max_{m\in[m_1,m_2]} I_1(c,k,m), \tag{5.6}$$

$$\max_{m\in[m_1,m_2]} I_2(c_0,k_0,m) \leq U. \tag{5.7}$$

5.1.0.2 Problem 5.2. Find parameters $c^0 \geq 0$ and $k^0 \geq 0$ such that

$$\max_{m\in[m_1,m_2]} I_2(c^0,k^0,m) = \min_{c\geq 0, k\geq 0}\max_{m\in[m_1,m_2]} I_2(c,k,m), \tag{5.8}$$

$$\max_{m\in[m_1,m_2]} I_1(c^0,k^0,m) \leq D. \tag{5.9}$$

In Problem 5.1, the guaranteed minimum of the peak displacement of the body to be isolated is sought, with a constraint being imposed on its absolute acceleration. In Problem 5.2, which is the reciprocal of Problem 5.1, the guaranteed minimum of the peak acceleration is sought with a constraint on the displacement of the body to be isolated.

Note that the maximum of the function $I_2(c,k,m)$ of Eq. (5.5) with respect to m expresses the maximum (with respect to m) of the peak acceleration of the body being isolated. The maximum of the peak acceleration of the body and the maximum of the peak transmitted force correspond to different values of m. As a consequence, unlike the case of a fixed m, Problem 5.2 is not equivalent to the problem of maximization of the peak force transmitted to the body being isolated. Also, the constraint of Eq. (5.7) on the maximum (with respect to m) of the peak acceleration is not equivalent to the constraint on the maximum (with respect to m) of the peak force transmitted to the body.

5.1.0.3 Example 5.1. *Maxima of the Peak Force and the Peak Acceleration. Undamped Linear System.* Set $c = 0$ and $n = 1$ in Eq. (5.3). Then this equation becomes

$$m\ddot{x} + kx = 0, \quad x(0) = 0, \quad \dot{x}(0) = \beta. \tag{5.10}$$

The solution of (5.10) is given by

$$x(t) = \beta\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t. \tag{5.11}$$

Differentiate function (5.11) twice with respect to time to obtain

$$\ddot{x}(t) = \beta\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t, \tag{5.12}$$

which gives the peak acceleration

$$\max_t |\ddot{x}(t)| = \beta\sqrt{\frac{k}{m}} \tag{5.13}$$

and

$$\max_{m\in[m_1,m_2]}\max_t |\ddot{x}(t)| = \beta\sqrt{\frac{k}{m_1}}. \tag{5.14}$$

On the other hand, according to (5.10), the force acting on the body is equal to $-kx$. Accordingly, the peak magnitude of the force is

$$\max_t |kx(t)| = \max_t \left|\beta\sqrt{mk}\sin\sqrt{\frac{k}{m}}t\right| = \beta\sqrt{mk} \tag{5.15}$$

and

$$\max_{m\in[m_1,m_2]}\max_t |kx(t)| = \beta\sqrt{m_2 k}. \tag{5.16}$$

It is evident from (5.14) and (5.16) that the maximum of the peak acceleration occurs at $m = m_1$, whereas the maximum of the peak force occurs at $m = m_2$.

5.2 ANALYSIS OF THE OPTIMIZATION PROBLEM.

5.2.1 Formulation of the Problem in Dimensionless Variables.

Begin by solving Problem 5.2, where the performance index to be minimized is the peak acceleration of the body to be isolated, while the peak displacement of the body is subject to a constraint. Recall that we are considering the case where the system is subjected to an instantaneous impact from which the body receives an initial velocity β. Introduce the dimensionless variables and parameters

$$x' = \frac{x\,\mathrm{sign}(\beta)}{D}, \quad t' = \frac{t|\beta|}{D}, \quad c' = \frac{cD|\beta|^{r-2}}{m_2}, \quad k' = \frac{kD^{n+1}}{m_2\beta^2}, \tag{5.17}$$

$$\alpha = \frac{m_2}{m}, \quad \alpha_0 = \frac{m_2}{m_1}.$$

The value of the dimensionless parameter α changes as the mass m of the body changes. Since m belongs to the interval $m_1 \leq m \leq m_2$, the dimensionless parameter α falls in the interval $1 \leq \alpha \leq \alpha_0$. Substitute the expressions of Eq. (5.17) into Eq. (5.3) to obtain

$$\ddot{x} + \alpha c|\dot{x}|^r \operatorname{sign}(\dot{x}) + \alpha k|x|^n \operatorname{sign}(x) = 0, \quad r \geq 2, \quad n \geq 1, \tag{5.18}$$

$$x(0) = 0, \quad \dot{x}(0) = 1, \quad \alpha \in [1, \alpha_0].$$

In Eq. (5.18) and in what follows, the primes indicating the dimensionless variables are omitted. In the new variables, Problem 5.2 is reformulated as

5.2.1.1 Problem 5.2a. Find $c^0 \geq 0$ and $k^0 \geq 0$ such that

$$\max_{\alpha \in [1,\alpha_0]} I_2(c^0, k^0, \alpha) = \min_{c \geq 0, k \geq 0} \max_{\alpha \in [1,\alpha_0]} I_2(c, k, \alpha), \tag{5.19}$$

$$\max_{\alpha \in [1,\alpha_0]} I_1(c^0, k^0, \alpha) \leq 1. \tag{5.20}$$

Here,

$$I_1(c, k, \alpha) = \max_{t \in [0,\infty)} |x(t)|, \tag{5.21}$$

$$I_2(c, k, \alpha) = \alpha \max_{t \in [0,\infty)} |c|\dot{x}(t)|^r \operatorname{sign}(\dot{x}(t)) + k|x(t)|^n \operatorname{sign}(x(t))|, \tag{5.22}$$

where $x(t)$ denotes the solution of the initial value problem of Eq. (5.18). Note that the optimization problem defined by Eqs. (5.18), (5.19), and (5.20) contains only one parameter, $\alpha_0 = m_2/m_1$, to characterize the uncertainty in the magnitude of the mass of the body to be isolated.

5.2.2 Preliminary Consideration.

According to Proposition 2.3 in Chapter 2, the function $I_1(c, k, \alpha)$ monotonically decreases as the products αc and αk increase. Hence, the criterion $I_1(c, k, \alpha)$ monotonically decreases as the variable α increases, and, therefore,

$$\max_{\alpha \in [1,\alpha_0]} I_1(c, k, \alpha) = I_1(c, k, 1). \tag{5.23}$$

Physically, this result is expected. For prescribed initial velocity and isolator parameters, the greater the mass of the body to be isolated, the greater the maximum of its displacement.

Since we assumed $r \geq 2$ and $n \geq 1$ in Eqs. (5.3) and (5.18), the function $I_2(c, k, \alpha)$ can be represented as

$$I_2(c, k, \alpha) = \alpha \max\{c, k[I_1(c, k, \alpha)]^n\}, \quad \alpha \in [1, \alpha_0], \tag{5.24}$$

according to the results of Section 2.3 of Chapter 2. In this case, the maximum value of the absolute acceleration of the body to be isolated occurs either at the initial time instant or at the instant at which the body's relative velocity vanishes for the first time. According to Eqs. (5.23) and (5.24), Problem 5.2a can be reformulated as

5.2.2.1 Problem 5.2b. Find the optimal parameters c^0 and k^0 such that

$$\max_{\alpha \in [1, \alpha_0]} \max\{\alpha c^0, \alpha k^0 [I_1(c^0, k^0, \alpha)]^n\} = \min_{c,k} \max_{\alpha \in [1, \alpha_0]} \max\{\alpha c, \alpha k [I_1(c, k, \alpha)]^n\}, \tag{5.25}$$

provided the constraint

$$I_1(c^0, k^0, 1) \leq 1 \tag{5.26}$$

is satisfied.

It can be shown that the optimal parameters lie on the curve

$$\Gamma_1 = \{c, k : I_1(c, k, 1) = 1\}. \tag{5.27}$$

To prove this fact, it is sufficient to show that the parameters c and k minimizing the criterion $I_2(c, k, \alpha)$ of Eq. (5.24), under the constraint of Eq. (5.26), for any fixed α, belong to the curve Γ_1 of Eq. (5.27). The proof of the latter property virtually coincides with the proof of Proposition 2.7 in Chapter 2.

According to Proposition 2.3, the criterion $I_1(c, k, 1)$ monotonically decreases with respect to c and k. Hence, by the implicit function theorem, the equation $I_1(c, k, 1) = 1$ specifying the curve Γ_1, can be represented as a monotone decreasing function

$$k = k_1(c). \tag{5.28}$$

Substitute the variable k of Eq. (5.28) into Eq. (5.24) to obtain the function

$$\theta(c, \alpha) = I_2(c, k_1(c), \alpha) = \max\{\alpha c, \alpha k_1(c)[I_1(c, k_1(c), \alpha)]^n\}, \tag{5.29}$$

which represents the criterion $I_2(c, k, \alpha)$ on the curve Γ_1. Since the optimal parameters c^0 and k^0 belong to the curve Γ_1, to solve Problem 5.2b one should find the optimal parameter c^0 from the minmax condition

$$\max_{\alpha\in[1,\alpha_0]} \theta(c^0,\alpha) = \min_{c\geq 0} \max_{\alpha\in[1,\alpha_0]} \theta(c,\alpha) \tag{5.30}$$

for the function $\theta(c,\alpha)$. Then the optimal parameter k^0 is determined by

$$k^0 = k_1(c^0). \tag{5.31}$$

With allowance for the properties of the function $k_1(c)$ of Eq. (5.28) and the function $\theta(c,\alpha)$ of Eq. (5.29), we can find an interval $[c_1, c_2] \neq [0,\infty)$ to which the optimal parameter c^0 belongs. This simplifies the solution of the minmax problem of Eq. (5.30).

Consider the curve

$$B^\alpha = \{c, k : c = k[I_1(c,k,\alpha)]^n\} \tag{5.32}$$

dividing, for a fixed α, the first quadrant of the ck-plane into two regions (Fig. 5.2). To the right of and below the curve B^α, the criterion $I_2(c,k,\alpha)$ is represented as

$$I_2(c,k,\alpha) = \alpha c \tag{5.33}$$

and the inequality

$$c > k[I_1(c,k,\alpha)]^n \tag{5.34}$$

is satisfied. To the left of and above the curve B^α,

$$I_2(c,k,\alpha) = \alpha k[I_1(c,k,\alpha)]^n. \tag{5.35}$$

and the inequality

$$c < k[I_1(c,k,\alpha)]^n \tag{5.36}$$

is satisfied. The curve B^α has properties similar to those of the curve B given by Eq. (2.123). In particular, the curve B^α can be specified as a function

$$k = k_B(c,\alpha) \tag{5.37}$$

which monotonically increases with respect to c, and, in addition, $k_B(0,\alpha) = 0$. According to the definition of the curve B^α (Eq. (5.32)), the function $k_B(c,\alpha)$ is defined implicitly by the equation

$$c = k[I_1(c,k,\alpha)]^n. \tag{5.38}$$

In Section 2.3.4 of Chapter 2, it was established that the optimal parameters in Problem 2.4, for isolator characteristics specified by Eq. (2.112) with $r \geq 2$ and $n \geq 1$, are the coordinates of the

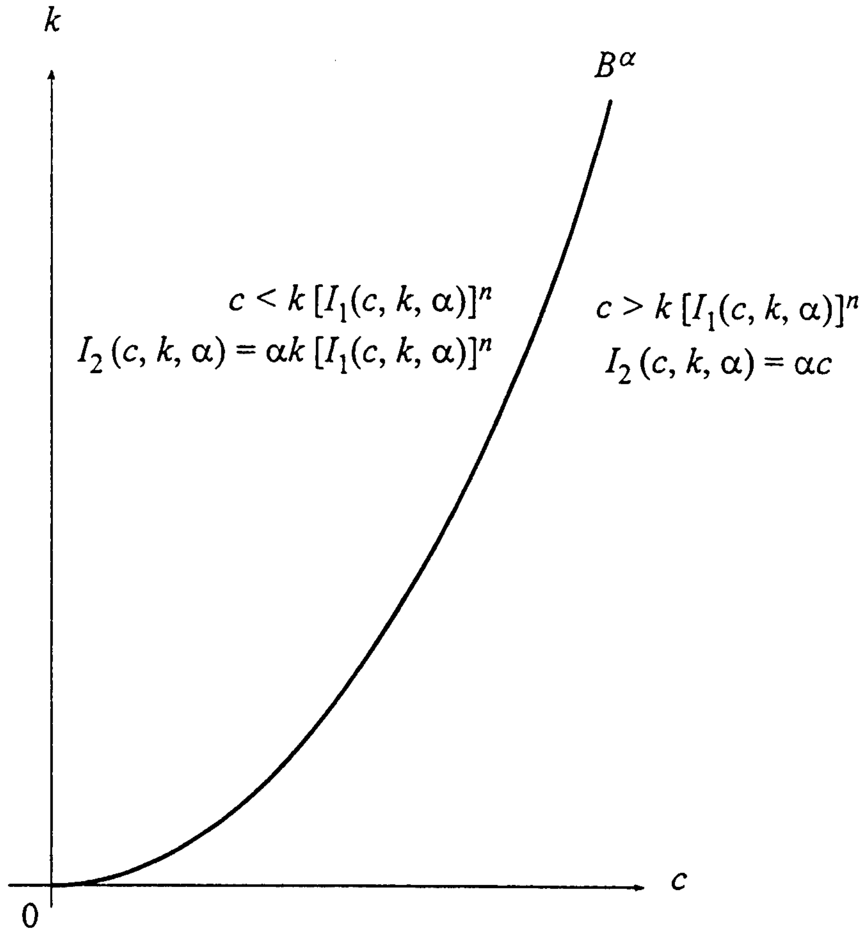


Figure 5-2. General view of the curve B^{α}.

point of the intersection of the curve B and the curve Γ_1 specified by Eq. (2.117). By using arguments similar to those employed in Section 2.3.4 to prove the analogous property, one can show that, for a fixed α, the parameters c and k minimizing the criterion of Eq. (5.24) under the constraint of Eq. (5.26) are the coordinates of the point of the intersection of the curve B^α with the curve Γ_1 of Eq. (5.27). Hence, the optimal point corresponding to the optimal parameters c^0 and k^0 in Problem 5.2b (and, consequently, in Problem 5.2a) should be sought among the points of intersection of the curve Γ_1 with the curves B^α for $\alpha \in [1, \alpha_0]$.

Denote by $c^*(\alpha)$ the value of the parameter c corresponding to the point of the intersection of the curves B^α and Γ_1. The function $c^*(\alpha)$ is illustrated in Fig. 5.3. Since the curve B^α is represented by Eq. (5.32) and the curve Γ_1 is represented by Eq. (5.27), the function $c^*(\alpha)$ is defined implicitly by the equation

$$k_B(c, \alpha) - k_1(c) = 0. \tag{5.39}$$

5.2.2.2 Lemma 5.1. The function $c^*(\alpha)$ monotonically decreases.

5.2.2.3 Proof. To prove this lemma, let us show that the left-hand side of Eq. (5.39) monotonically increases with respect to both variables c and α. Then, according to the implicit function theorem, the function $c^*(\alpha)$ will be monotonically decreasing. The monotone increase of the left-hand side of Eq. (5.39) with respect to c follows from the monotone increase of the function $k_B(c, \alpha)$ with respect to c and the monotone decrease of the function $k_1(c)$. Now, let us show that the function $k_B(c, \alpha)$ monotonically increases with respect to α. Calculate the partial derivative of $k_B(c, \alpha)$ with respect to α using Eq. (5.38), which defines the function $k_B(c, \alpha)$ implicitly. This yields

$$\frac{\partial k_B(c, \alpha)}{\partial \alpha} = -nk[I_1(c, k, \alpha)]^{n-1} \frac{\partial I_1(c, k, \alpha)}{\partial \alpha} \left\{ \frac{\partial [k(I_1(c, k, \alpha))^n]}{\partial k} \right\}^{-1}. \tag{5.40}$$

The derivative $\partial I_1(c, k, \alpha)/\partial \alpha$ is negative, according to Proposition 2.3. The expression in the braces on the right-hand side of Eq. (5.40) is positive, according to Lemma 2.1. Thus, the right-hand side in Eq. (5.40) is positive, and, hence, function $k_B(c, \alpha)$ monotonically increases with respect to α. It is evident from Eq. (5.39) that the left-hand side of this equation also monotonically increases with respect to α.

This completes the proof of the lemma.

Figure 5.4 illustrates the relative position of the curve Γ_1 and the curves B^α on the ck-plane. Since the function $k_B(c, \alpha)$ monotonically increases with respect to α, all curves B^α lie between the curves $B^1 = \{c, k : k = k_B(c, 1)\}$ and $B^{\alpha_0} = \{c, k : k = k_B(c, \alpha_0)\}$. Moreover, the larger the value of α, the higher the position of the curve B^α.

It follows from Lemma 5.1 and Fig. 5.4 that the optimal parameter c^0 in Problem 5.2b belongs to the interval $[c_1, c_2]$, where

$$c_1 = c^*(\alpha_0), \quad c_2 = c^*(1). \tag{5.41}$$

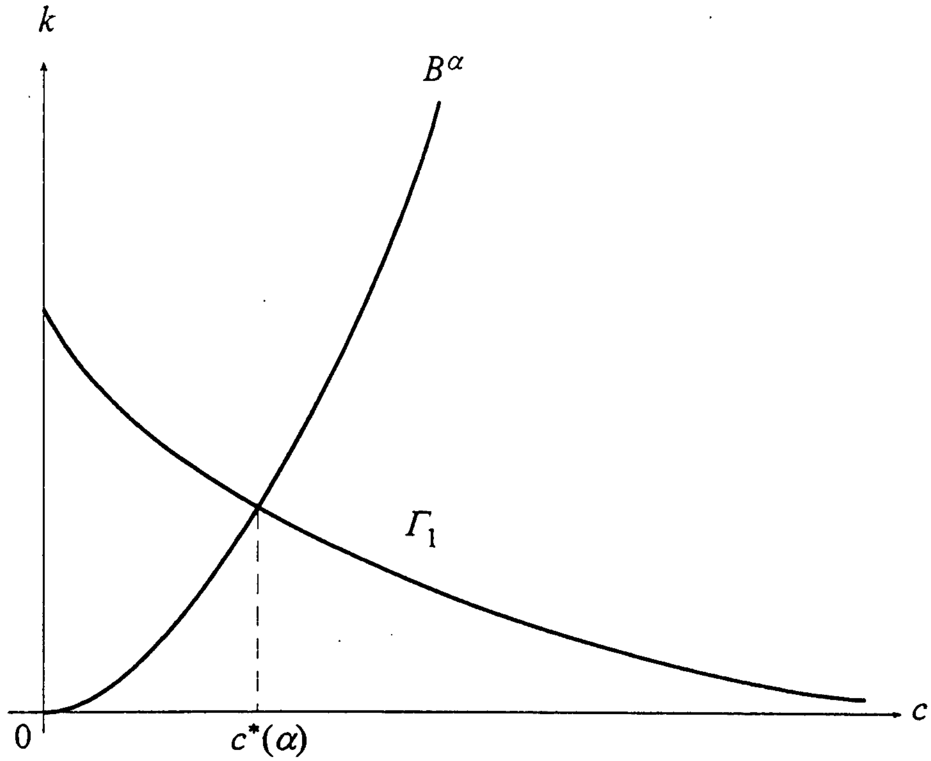


Figure 5-3. Definition of the function $c^*(\alpha)$.

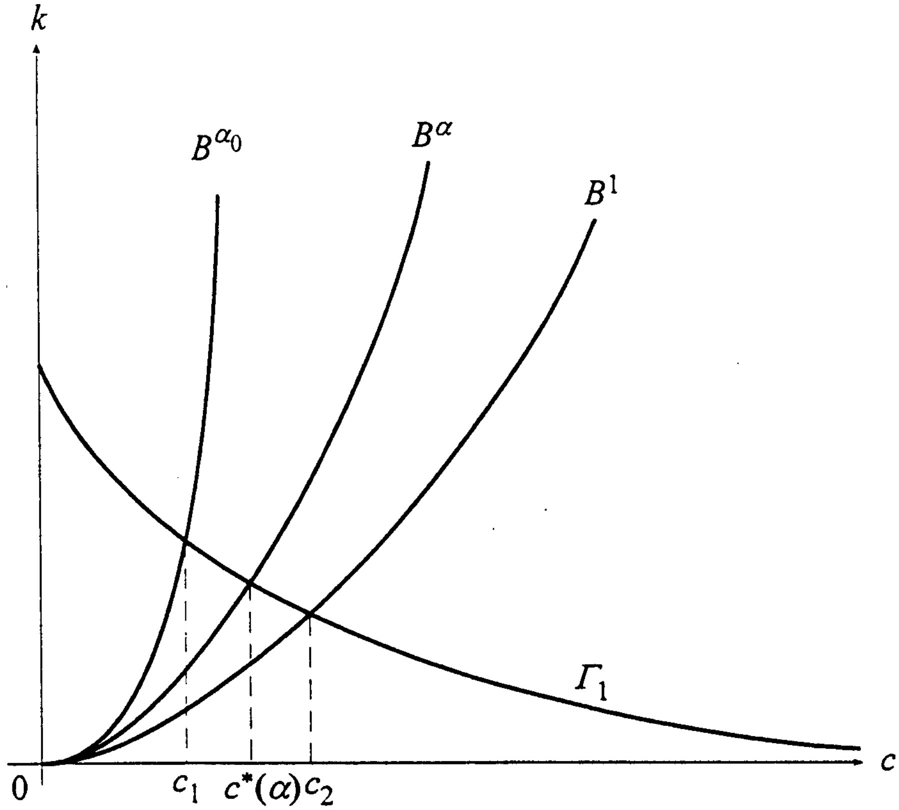


Figure 5-4. Relative position of the curves Γ_1 and B^{α} on the ck-plane. The parameter α lies in the interval $1 < \alpha < \alpha_0$, $c_1 = c^*(\alpha_0)$, $c_2 = c^*(1)$.

Thus, the minmax condition of Eq. (5.30) can be rewritten as

$$\max_{\alpha\in[1,\alpha_0]} \theta(c^0,\alpha) = \min_{c\in[c_1,c_2]} \max_{\alpha\in[1,\alpha_0]} \theta(c,\alpha). \tag{5.42}$$

From the definition of the function $\theta(c,\alpha)$ in Eq. (5.29), it follows that

$$\max_{\alpha\in[1,\alpha_0]} \theta(c,\alpha) = \max\{\max_{\alpha\in[1,\alpha_0]} \alpha c, \max_{\alpha\in[1,\alpha_0]} \{\alpha k_1(c)[I_1(c,k_1(c),\alpha)]^n\}\}. \tag{5.43}$$

Introduce the notation

$$\gamma = \gamma(c,\alpha) = \alpha k_1(c)[I_1(c,k_1(c),\alpha)]^n, \tag{5.44}$$

$$\sigma(c) = \max_{\alpha\in[1,\alpha_0]} \gamma(c,\alpha). \tag{5.45}$$

Then Eq. (5.43) becomes

$$\max_{\alpha\in[1,\alpha_0]} \theta(c,\alpha) = \max\{\alpha_0 c, \sigma(c)\}. \tag{5.46}$$

5.2.2.4 Lemma 5.2. The function $\sigma(c)$ of Eq. (5.45) monotonically decreases.

5.2.2.5 Proof. The function $\gamma(c,\alpha)$ of Eq. (5.44) monotonically decreases with respect to c for any α. Indeed, according to Lemma 2.1, the function $k[I_1(c,k,\alpha)]^n$ monotonically increases with respect to k. According to Proposition 2.3, the function $I_1(c,k,\alpha)$ monotonically decreases with respect to c. As was mentioned above, the function $k_1(c)$ monotonically decreases with respect to c. The monotone decrease of the function $\gamma(c,\alpha)$ with respect to c follows from the cited properties of the functions $k[I_1(c,k,\alpha)]^n$, $I_1(c,k,\alpha)$, and $k_1(c)$. Since the function $\gamma(c,\alpha)$, which is maximized with respect to α in Eq. (5.45), monotonically decreases with respect to c for any α, the function $\sigma(c)$ decreases monotonically.

Denote the right-hand side of Eq. (5.46) by $\xi(c)$, i.e.,

$$\xi(c) = \max\{\alpha_0 c, \sigma(c)\} \tag{5.47}$$

and investigate the function $\xi(c)$ on the interval $[c_1, c_2]$. Let us show that either

$$\xi(c) = \alpha_0 c \quad \text{for} \quad c \in [c_1, c_2] \tag{5.48}$$

or

$$\xi(c) = \begin{cases} \sigma(c), & \text{for } c \in [c_1, \bar{c}) \\ \alpha_0 c, & \text{for } c \in [\bar{c}, c_2] \end{cases} . \tag{5.49}$$

The function $\alpha_0 c$ monotonically increases. The function $\sigma(c)$ monotonically decreases, according to Lemma 5.2. These properties imply three possibilities for the behavior of the function $\xi(c)$ of Eq. (5.47).

Case 1. If $\alpha_0 c_1 \geq \sigma(c_1)$, then the line $\alpha_0 c$ lies above the curve $\sigma(c)$ on the interval $(c_1, c_2]$, and $\xi(c) = \alpha_0 c$ for $c \in [c_1, c_2]$. See Fig. 5.5.

Case 2. If $\alpha_0 c_2 \leq \sigma(c_2)$, then the line $\alpha_0 c$ lies below the curve $\sigma(c)$ on the interval $[c_1, c_2)$, and $\xi(c) = \sigma(c)$ for $c \in [c_1, c_2]$. See Fig. 5.6.

Case 3. If neither case 1 nor case 2 occur, i.e., the inequalities $\alpha_0 c_1 < \sigma(c_1)$ and $\alpha_0 c_2 > \sigma(c_2)$ hold simultaneously, then the line $\alpha_0 c$ and the curve $\sigma(c)$ intersect at some point $\bar{c} \in (c_1, c_2)$. See Fig. 5.7.

Case 1 corresponds to Eq. (5.48). Case 3 corresponds to Eq. (5.49). Case 2 is impossible, since the inequality

$$\alpha_0 c_2 > \sigma(c_2) \tag{5.50}$$

holds. Indeed , the curve B^1 lies below all the other curves B^α, $\alpha \in (1, \alpha_0]$, as shown in Fig.5.4. The point with the coordinates $c = c_2$ and $k = k_1(c_2)$ belongs to the curve B^1 and, hence, according to Eq. (5.32),

$$c_2 = k_1(c_2)[I_1(c_2, k_1(c_2), 1)]^n. \tag{5.51}$$

Since the point with the coordinates $c = c_2$ and $k = k_1(c_2)$ belongs to the curve B^1, it lies below all the curves B^α, $\alpha \in (1, \alpha_0]$, and according to Eq. (5.36),

$$c_2 > k_1(c_2)[I_1(c_2, k_1(c_2), \alpha)]^n, \quad \alpha \in (1, \alpha_0]. \tag{5.52}$$

Multiply Eq. (5.52) by α to obtain

$$\alpha c_2 > \alpha k_1(c_2)[I_1(c_2, k_1(c_2), \alpha)]^n, \quad \alpha \in (1, \alpha_0]. \tag{5.53}$$

With allowance for Eq. (5.44), this inequality can be rewritten as

$$\alpha c_2 > \gamma(c_2, \alpha), \quad \alpha \in (1, \alpha_0]. \tag{5.54}$$

Since $\alpha_0 \geq \alpha$, Eq. (5.54) implies

$$\alpha_0 c_2 > \gamma(c_2, \alpha), \quad \alpha \in (1, \alpha_0]. \tag{5.55}$$

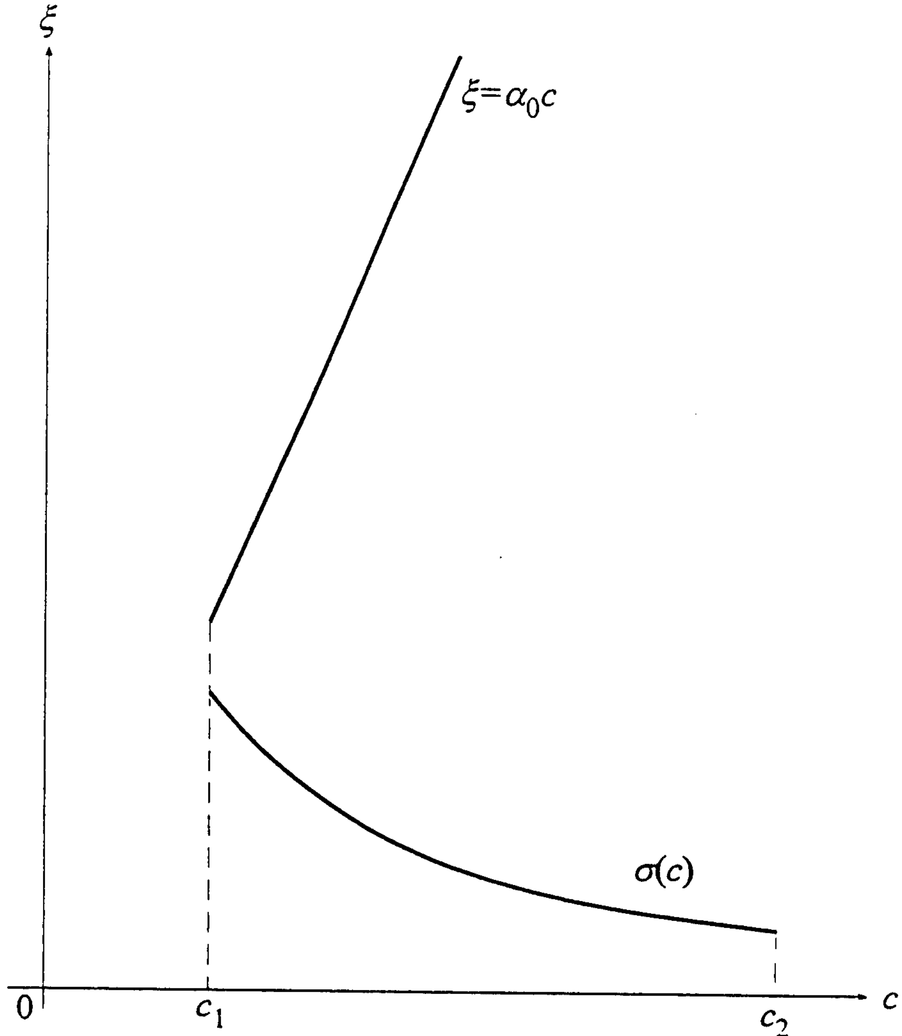


Figure 5-5. The function $\xi(c)$ for the case where $\alpha_0 c_1 \geqslant \sigma(c_1)$.

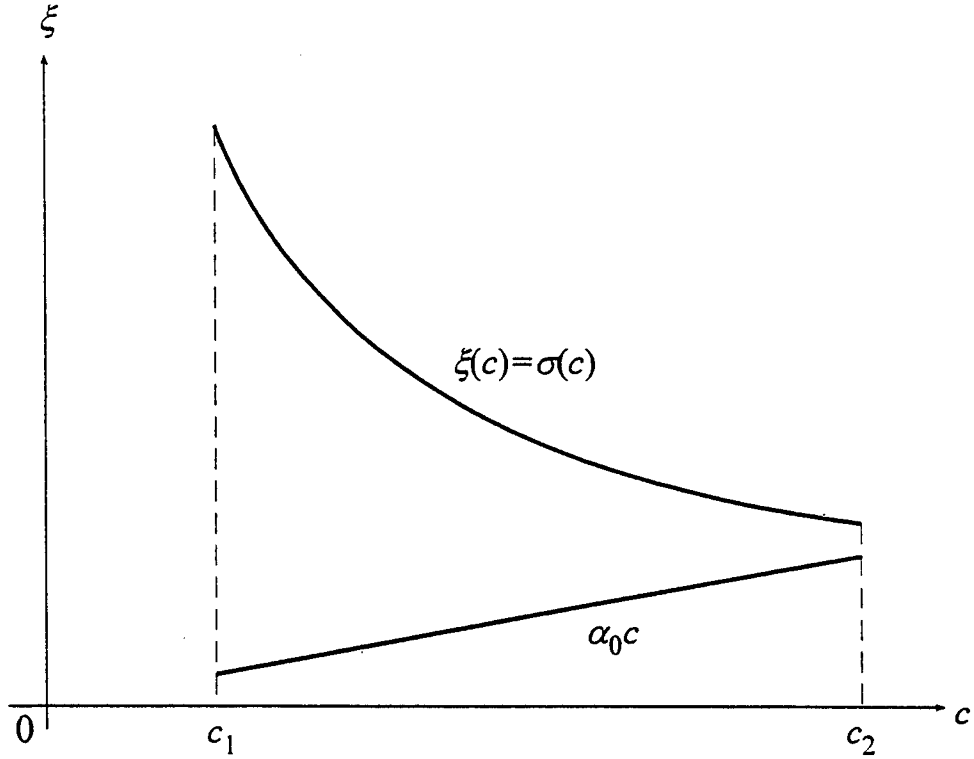


Figure 5-6. The function $\xi(c)$ for the case where $\alpha_0 c_2 \leqslant \sigma(c_2)$.

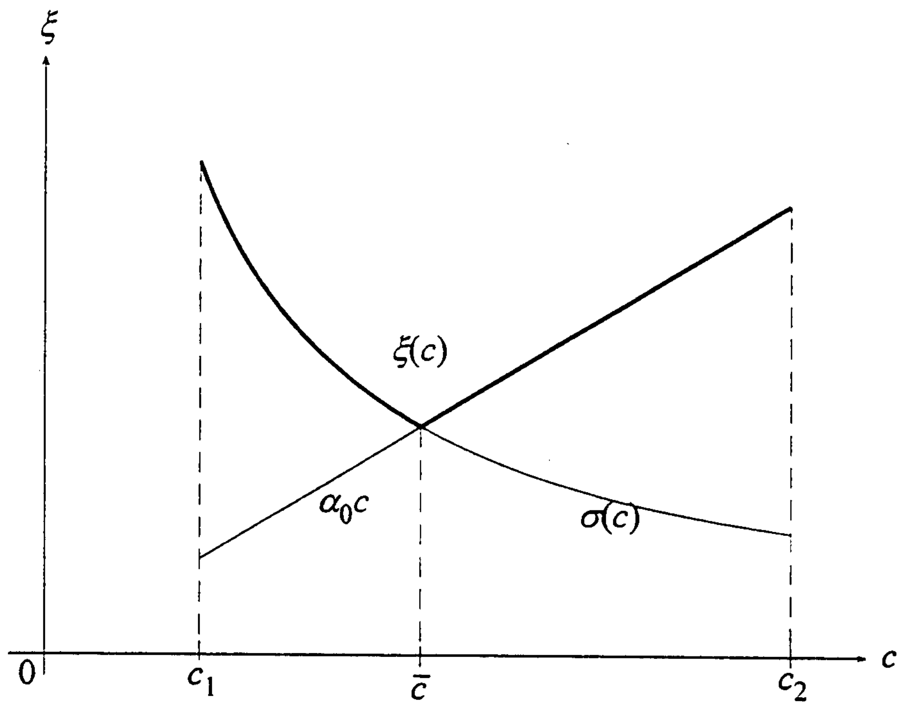


Figure 5-7. The function $\xi(c)$ for the case where $\alpha_0 c_1 < \sigma(c_1)$ and $\alpha_0 c_2 > \sigma(c_2)$.

According to Eqs. (5.44) and (5.51), $\gamma(c_2, 1) = c_2$, and, since $\alpha_0 > 1$, we have

$$\alpha_0 c_2 > \gamma(c_2, 1). \tag{5.56}$$

Equations (5.55) and (5.56), combined with the definition of the function $\sigma(c)$ in Eq. (5.45), imply Eq. (5.50).

It is evident that the minimum of the function $\xi(c)$ is provided by the point $c = c_1$ for the case of Eq. (5.48), and by the point $\bar{c}$ for the case of Eq. (5.49).

5.2.3 Algorithm for Calculating the Optimal Parameters.

An algorithm for solving Problem 5.2b can now be outlined. To find the desired optimal parameters c^0 and k^0, one can follow the steps:

Step 1. Calculate the function $k_1(c)$ of Eq. (5.28) by solving numerically the equation $I_1(c, k, 1) = 1$ with respect to k.

Step 2. Solve the equation $c = k_1(c)[I_1(c, k_1(c), \alpha_0)]^n$ for c to find $c_1 = c^*(\alpha_0)$.

Step3. Solve the equation $c = k_1(c)[I_1(c, k_1(c), 1)]^n$ for c to find $c_1 = c^*(1)$. Note that, according to the definition of the curve Γ_1 in Eq. (5.27), $I_1(c, k_1(c), 1) = 1$, and the equation $c = k_1(c)[I_1(c, k_1(c), 1)]^n$ becomes $c = k_1(c)$.

Step 4. Calculate $\sigma(c_1)$ for the function σ defined by Eq. (5.45).

Step 5. If $\sigma(c_1) \leq \alpha_0 c_1$, then the desired optimal parameters are given by

$$c^0 = c_1, \quad k^0 = k_1(c_1). \tag{5.57}$$

Step 6. If $\sigma(c_1) > \alpha_0 c_1$, then solve the equation

$$\alpha_0 c = \sigma(c) \tag{5.58}$$

for c to find the point $\bar{c}$ in Eq. (5.49). The desired optimal parameters are given by

$$c^0 = \bar{c}, \quad k^0 = k_1(\bar{c}). \tag{5.59}$$

Step 7. Calculate the optimal value of the peak acceleration of the body being isolated as

$$\max_{\alpha \in [1, \alpha_0]} I_2(c^0, k^0, \alpha) = \xi(c^0) \tag{5.60}$$

where the function $\xi(c)$ is defined in Eq. (5.47).

5.2.4 Solutions of the Optimization Problems in the Original Variables.

5.2.4.1 <u>Problem 5.2</u>. In the previous section, we presented an algorithm for the solution of problem 5.2b, which is Problem 5.2 represented in the dimensionless variables of Eq. (5.17). We can return to the original variables to obtain the formulas for the optimal parameters of the isolator and the corresponding values of the performance criteria. Thus, we arrive at the final solution of Problem 5.2

$$c^0 = \frac{m_2|\beta|^{2-r}}{D}(c^0)', \quad k^0 = \frac{m_2\beta^2}{D^{n+1}}(k^0)' \tag{5.61}$$

$$\max_{m\in[m_1,m_2]} I_2(c^0, k^0, m) = \frac{\beta^2}{D} \max_{\alpha\in[1,\alpha_0]} I_2'[(c^0)', (k^0)', \alpha], \tag{5.62}$$

$$\max_{m\in[m_1,m_2]} I_1(c^0, k^0, m) = D, \tag{5.63}$$

where primes indicate the corresponding quantities represented in the dimensionless variables.

5.2.4.2 <u>Problem 5.1</u>. It is evident from Eq. (5.62) that the optimal value of the criterion to be minimized in Problem 5.2 monotonically decreases as the variable D, characterizing the constraint, increases. Also, as follows from Eqs. (5.63) and (5.9), the optimum occurs on the boundary of the admissible set of design variables specified by Eq. (5.9). Hence, Problems 5.1 and 5.2 are dual to each other, in the sense of Theorem 1.1. Apply Theorem 1.1 to the solution of Problem 5.2, given by Eqs. (5.61) to (5.63), to arrive at the solution to Problem 5.1:

$$c_0 = \frac{m_2 U}{|\beta|^r}\left\{\max_{\alpha\in[1,\alpha_0]} I_2'[(c^0)', (k^0)', \alpha]\right\}^{-1}(c^0)', \tag{5.64}$$

$$k_0 = \frac{m_2 U^{n+1}}{\beta^{2n}}\left\{\max_{\alpha\in[1,\alpha_0]} I_2'[(c^0)', (k^0)', \alpha]\right\}^{-(n+1)}(k^0)', \tag{5.65}$$

$$\max_{m\in[m_1,m_2]} I_1(c_0, k_0, m) = \frac{\beta^2}{U} \max_{\alpha\in[1,\alpha_0]} I_2'[(c^0)', (k^0)', \alpha], \tag{5.66}$$

$$\max_{m\in[m_1,m_2]} I_2(c_0, k_0, m) = U. \tag{5.67}$$

5.3 OPTIMAL PARAMETERS FOR THE ISOLATOR WITH A LINEAR SPRING AND A QUADRATIC DAMPER.

5.3.1 Preliminary Analysis.

Turn now to the problem of finding the optimal parameters of the isolator with a linear spring and a quadratic damper. This corresponds to $n = 1$ and $r = 2$ in Eqs. (5.3) and (5.18). For this

isolator, Eq. (5.18), which governs the motion of the body being isolated when subjected to an instantaneous impact, becomes

$$\ddot{x} + \alpha c\dot{x}|\dot{x}| + \alpha kx = 0, \tag{5.68}$$

$$x(0) = 0, \quad \dot{x}(0) = 1, \quad \alpha \in [1, \alpha_0].$$

Recall that $\alpha = m_2/m$, and $\alpha_0 = m_2/m_1$, where m is the incompletely prescribed mass of the body to be isolated, and m_1 and m_2 are, respectively, the lower and upper bounds for the range of m. Thus, as m ranges through the interval $m_1 \leq m \leq m_2$, the dimensionless parameter α ranges through the interval $1 \leq \alpha \leq \alpha_0$.

According to Propositions 2.1 and 2.2 of Chapter 2, both the peak displacement and the peak acceleration of the body occur on the interval $0 \leq t \leq t_*$, where t_* is the instant at which the velocity $\dot{x}$ vanishes for the first time. On the interval $0 \leq t < t_*$, the velocity is positive ($\dot{x} > 0$) and, hence, the coordinate x monotonically increases and is nonnegative. Therefore, we can take x as a new independent variable. This reduces the second-order equation (Eq. 5.68) to the first-order equation for the variable $y = \dot{x}$

$$y\frac{dy}{dx} + \alpha cy^2 + \alpha kx = 0, \qquad y(0) = 1. \tag{5.69}$$

Introduce the variable $w = y^2$ to reduce the nonlinear Eq. (5.69) to the linear equation

$$\frac{dw}{dx} + 2\alpha cw = -2\alpha kx, \qquad w(0) = 1. \tag{5.70}$$

The solution to the initial-value problem of Eq. (5.70) is given by

$$w(x) = \left\{1 - \frac{k}{2\alpha c^2}\left[1 - \exp(2\alpha cx)(1 - 2\alpha cx)\right]\right\}\exp(-2\alpha cx), \quad \text{if } c \neq 0 \tag{5.71}$$

$$w(x) = 1 - \alpha kx^2, \quad \text{if } c = 0. \tag{5.72}$$

Let us show that for the system described by Eq. (5.68), the relationship

$$\sigma(c_1) = \alpha_0 c_1 \tag{5.73}$$

holds and, hence, according to Step 5 of the algorithm presented in Section 5.2.3, the dimensionless optimal parameters are determined by Eq. (5.57). Here, the function $\sigma(c)$ and the value c_1 of the parameter c are defined by Eqs. (5.45) and (5.41), respectively.

5.3.1.1 <u>Lemma 5.3</u>. For the system of Eq. (5.68), the function $\alpha I_1(c, k, \alpha)$ monotonically increases with respect to the variable α for any $c > 0$ and $k > 0$.

5.3.1.2 Proof. Since the velocity y of the body being isolated vanishes at the point at which the maximum displacement occurs, the equation for $I_1 = I_1(c, k, \alpha)$ can be obtained by substituting $x = I_1$ into the right-hand side of Eq. (5.71) and setting the expression in braces to be equal to zero. This yields

$$1 - \frac{k}{2\alpha c^2}\left[1 - \exp(2\alpha c I_1)(1 - 2\alpha c I_1)\right] = 0. \tag{5.74}$$

Introduce the notation

$$z = \alpha I_1, \quad F(c, k, \alpha, z) = 1 - \frac{k}{2\alpha c^2}\left[1 - \exp(2cz)(1 - 2cz)\right] \tag{5.75}$$

to express Eq. (5.74) as

$$F(c, k, \alpha, z) = 0. \tag{5.76}$$

Equation (5.76) specifies the function $z(c, k, \alpha) = \alpha I_1(c, k, \alpha)$ implicitly. According to the implicit function theorem, the derivative $\partial z(c, k, \alpha)/\partial \alpha$ is expressed in terms of the function F as

$$\frac{\partial z(c, k, \alpha)}{\partial \alpha} = -\frac{\partial F(c, k, \alpha, z)}{\partial \alpha}\left[\frac{\partial F(c, k, \alpha, z)}{\partial z}\right]^{-1}. \tag{5.77}$$

Differentiation of the function $F(c, k, \alpha, z)$ of Eq. (5.75) with respect to α and z yields

$$\frac{\partial F}{\partial \alpha} = \frac{k}{2\alpha^2 c^2}\left[1 - \exp(2cz)(1 - 2cz)\right], \tag{5.78}$$

$$\frac{\partial F}{\partial z} = -\frac{2k}{\alpha} z \exp(2cz) \tag{5.79}$$

It is evident from Eq. (5.79) that $\partial F/\partial z < 0$ for $z > 0$. It can be shown that the expression in brackets in Eq. (5.78) is positive for $z > 0$. To prove this, note that the expression in brackets is equal zero for $z = 0$. Its derivative with respect to z is equal to $4c^2 z \exp(2cz)$ and, hence, is positive for $z > 0$. Thus, the expression in brackets in Eq. (5.78) monotonically increases as z increases and, hence, $\partial F/\partial \alpha > 0$ for $z > 0$. Since $\partial F/\partial \alpha > 0$ and $\partial F/\partial z < 0$, the expression of Eq. (5.77) implies

$$\frac{\partial z(c, k, \alpha)}{\partial \alpha} > 0. \tag{5.80}$$

Thus, the function $z(c, k, \alpha) = \alpha I_1(c, k, \alpha)$ monotonically increases with respect to α for any prescribed $c > 0$ and $k > 0$.

This completes the proof of the lemma.

It follows from Lemma 5.3 that the function $\gamma(c, \alpha)$ defined by Eq. (5.44) monotonically increases with respect to α for any $c > 0$. In turn, this property implies that the maximum of the function $\gamma(c, \alpha)$ over $\alpha \in [1, \alpha_0]$ occurs at $\alpha = \alpha_0$ for any c. Hence the function $\sigma(c)$ of Eq. (5.45) in the case in question, is given by

$$\sigma(c) = \gamma(c, \alpha_0) = \alpha_0 k_1(c) I_1(c, k_1(c), \alpha_0). \tag{5.81}$$

Now calculate $\sigma(c_1)$ according to Eq. (5.81). Recall that the function $k_1(c)$ specifies the curve Γ_1 defined by Eq. (5.27). The point with the coordinates c_1 and $k_1(c_1)$ is the point of intersection of the curve Γ_1 and the curve B^α, defined by Eq. (5.32), for $\alpha = \alpha_0$. According to Eq. (5.32), the relation $c = kI_1(c, k, \alpha)$ is valid on the curve B^α. Hence

$$\sigma(c_1) = \alpha_0 k_1(c_1) I_1(c_1, k_1(c_1), \alpha_0) = \alpha_0 c_1. \tag{5.82}$$

Thus, we have proved Eq. (5.73).

5.3.2 Calculation of the Optimal Parameters.

According to the algorithm presented in Section 5.2.3, if the relationship of Eq. (5.73) holds, then the optimal damping and stiffness coefficients are given by (see Eq. 5.57)

$$c^0 = c_1, \quad k^0 = k_1(c_1). \tag{5.83}$$

The point with the coordinates given by Eq. (5.83) is the point of intersection of the curve Γ_1, specified by the equation (Eq. 5.27)

$$I_1(c, k, 1) = 1, \tag{5.84}$$

and the curve B^{α_0}, specified by the equation (Eq. 5.32)

$$c = kI_1(c, k, \alpha_0). \tag{5.85}$$

First, let us show that $c_1 \neq 0$. It was established previously that Eq. (5.85) implicitly defines the monotonically increasing function $k = k_B(c, \alpha)$ satisfying the relation $k_B(0, \alpha) = 0$. Hence, $c = 0$ implies $k = 0$. However, for $c = 0$ and $k = 0$, the solution of Eq. (5.68) is given by $x(t) = t$. This solution is unbounded and, hence, the values $c = 0$ and $k = 0$ do not satisfy Eq. (5.84). Note also that $k_1(c_1) \neq 0$. It follows from the fact that the point with the coordinates c_1 and $k_1(c_1)$ is the point of intersection of the curves $k = k_1(c)$ and $k = k_B(c, \alpha_0)$. However, a point with $c \neq 0$ and $k = 0$ cannot belong to the curve $k = k_B(c, \alpha_0)$.

It was shown above that if $c > 0$, then the quantities $I_1(c, k, \alpha)$, k, c, and α are related by Eq. (5.74). Hence, by substituting $I_1 = 1$ and $\alpha = 1$ into Eq. (5.74) we obtain

$$1 - \frac{k}{2c^2}[1 - \exp(2c)(1 - 2c)] = 0, \tag{5.86}$$

which is equivalent to Eq. (5.84). Similarly, by substituting $I_1 = c/k$ and $\alpha = \alpha_0$ into Eq. (5.74), we obtain

$$1 - \frac{k}{2\alpha_0 c^2}\left[1 - \exp\left(\frac{2\alpha_0 c^2}{k}\right)\left(1 - \frac{2\alpha_0 c^2}{k}\right)\right] = 0, \tag{5.87}$$

which is equivalent to Eq. (5.85). Multiply Eq. (5.87) by $2\alpha_0 c^2/k$ and find

$$\left(\frac{2\alpha_0 c^2}{k} - 1\right)\left[1 - \exp\left(\frac{2\alpha_0 c^2}{k}\right)\right] = 0. \tag{5.88}$$

For $c \neq 0$ and $k \neq 0$, Eq. (5.88) implies

$$k = 2\alpha_0 c^2. \tag{5.89}$$

Substitution of Eq. (5.89) into Eq. (5.86) leads to

$$1 - \alpha_0[1 - \exp(2c)(1 - 2c)] = 0. \tag{5.90}$$

The solution of Eq. (5.90) for c is the desired value c_1.

Equation (5.90) has the unique solution $c = c_1$ and, moreover, $c_1 \in (0, 1/2)$. The uniqueness of the solution follows from the monotone decrease of the left-hand side of Eq. (5.90). The localization of the root of Eq. (5.90) within the interval $(0, 1/2)$ follows from the fact that the left-hand side of Eq. (5.90) is equal to 1 for $c = 0$ and is equal to $1 - \alpha_0 < 0$ for $c = 1/2$. Thus, to calculate the root of Eq. (5.90) for any $\alpha_0 > 1$ one can apply the bisection method on the interval $[0, 1/2]$.

According to Eq. (5.83), the value $c = c_1$ is equal to the optimal damping coefficient c^0. As follows from Eq. (5.90), the value of c^0 depends on the parameter α_0, i.e.,

$$c^0 = c^0(\alpha_0). \tag{5.91}$$

Figure 5.8 plots c^0 versus α_0.

From Eq. (5.89), the optimal stiffness coefficient is

$$k^0(\alpha_0) = 2\alpha_0[c^0(\alpha_0)]^2. \tag{5.92}$$

The value of k^0 is plotted in Fig. 5.9 as a function of α_0.

The optimal value of the peak acceleration of the body being isolated is calculated according to Eq. (5.60). For the isolator with a linear spring and a quadratic damper, the function $\xi(c)$ is given by Eq. (5.47) and, hence,

$$\max_{\alpha \in [1, \alpha_0]} I_2(c^0, k^0, \alpha) = \xi(c_1) = \xi(c^0(\alpha_0)) = \alpha_0 c^0(\alpha_0). \tag{5.93}$$

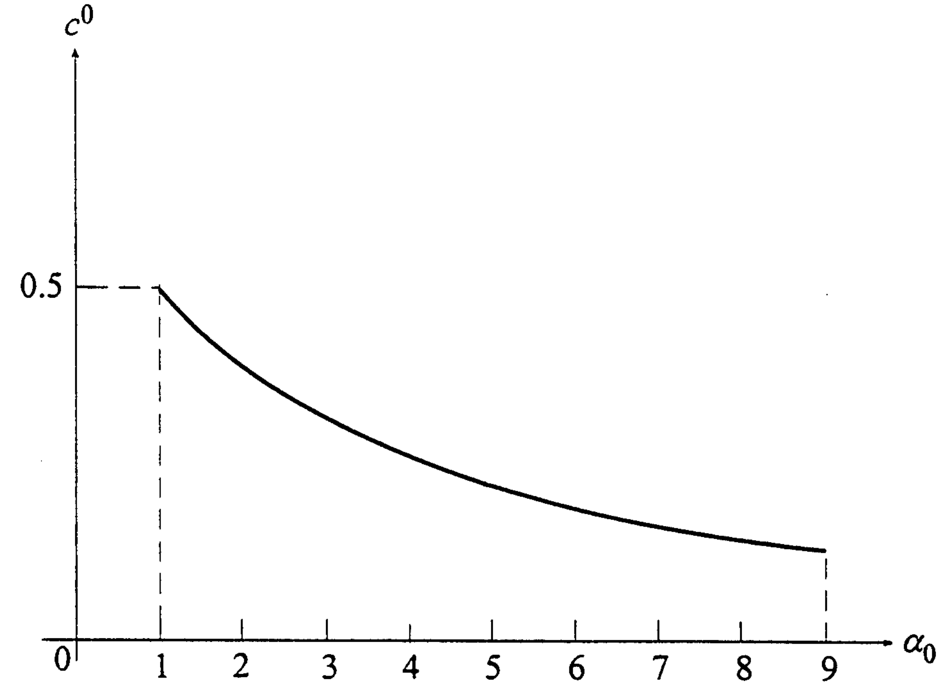


Figure 5-8. The dependence of the dimensionless optimal damping coefficient on the parameter α_0.

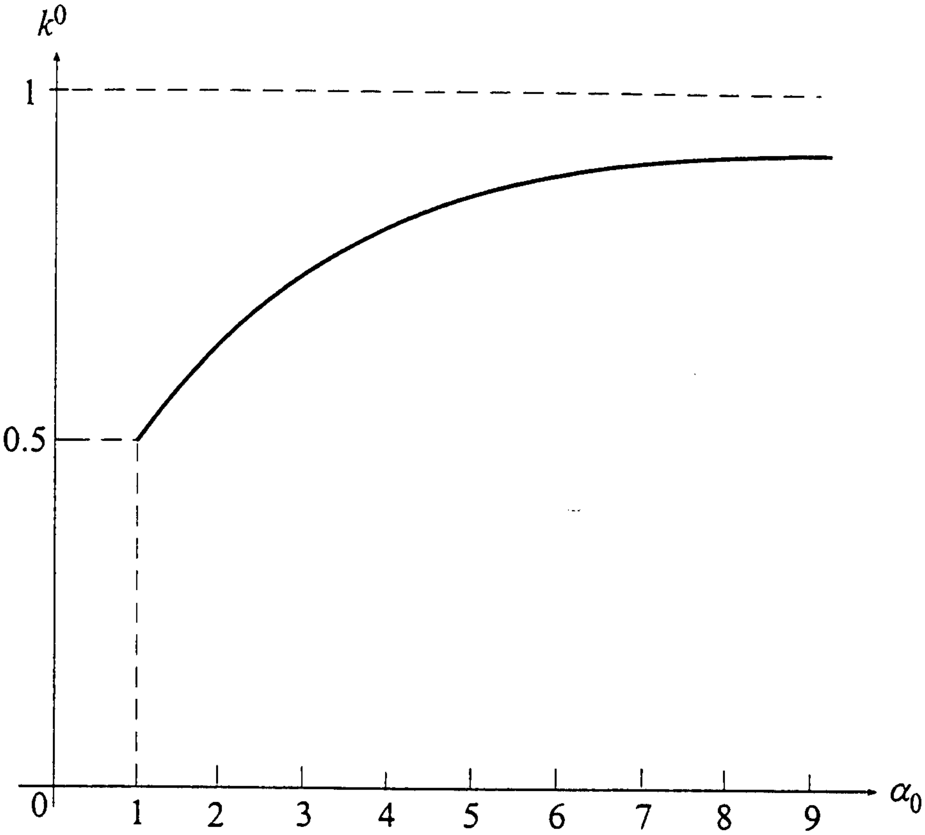


Figure 5-9. The dependence of the dimensionless optimal stiffness coefficient on the parameter α_0.

The optimal value of the peak acceleration of the body is plotted in Fig.5.10 as a function of α_0.

5.3.3 Asymptotic Behavior of the Solution for Large α_0.

Let us investigate the asymptotic behavior of the curves $c^0(\alpha_0)$, $k^0(\alpha_0)$, and $\alpha_0 c^0(\alpha_0)$ for large α_0, where $\alpha_0 = m_2/m_1$. As was mentioned above, the value $c^0(\alpha_0)$ is the root of Eq. (5.90) for c. Divide Eq. (5.90) by α_0 to obtain

$$1 - \exp(2c)(1 - 2c) = \frac{1}{\alpha_0}, \tag{5.94}$$

where $\alpha_0 = m_2/m_1$ is the ratio of the upper bound of the range of the mass of the body to be isolated to the lower bound of this range. As $\alpha_0 \to \infty$, Eq. (5.94) becomes

$$1 - \exp(2c)(1 - 2c) = 0. \tag{5.95}$$

It is readily verified that $c = 0$ is a root of Eq. (5.95). The derivatives of the left-hand sides of Eqs. (5.94) and (5.95) are equal to $4c\exp(2c)$. This final expression is positive for $c > 0$. Hence, Eqs. (5.94) and (5.95) have only one root. Since $c = 0$ is the root of Eq. (5.95), which is the limit case of Eq. (5.94) as $\alpha_0 \to \infty$, one can expect that for large α_0, the root of Eq. (5.94) will be close to zero. Expand the left-hand side of Eq. (5.94) into a Taylor series and retain only the first nonzero term. Then Eq. (5.94) is approximately

$$2c^2 \approx \frac{1}{\alpha_0}. \tag{5.96}$$

Solve Eq. (5.96) for c to obtain the desired asymptotic relation

$$c^0 \approx \sqrt{\frac{1}{2\alpha_0}}, \quad \alpha_0 \gg 1. \tag{5.97}$$

Substitute Eq. (5.97) into Eqs. (5.92) and (5.93) to find the asymptotic relations

$$k^0 \approx 1, \quad \alpha_0 \gg 1, \tag{5.98}$$

$$\max_{\alpha \in [1, \alpha_0]} I_2(c^0, k^0, \alpha) \approx \sqrt{\frac{\alpha_0}{2}}, \quad \alpha_0 \gg 1 \tag{5.99}$$

for the optimal stiffness coefficient and the maximum of the peak acceleration of the body being isolated, for large α_0.

It follows from Eqs. (5.97) to (5.99) that the dimensionless damping coefficient c^0 tends to zero as α_0 tends to infinity, the dimensionless stiffness coefficient k^0 tends to a finite limit, and the dimensionless optimal value of the performance index tends to infinity, as $\alpha_0 \to \infty$.

Recall that, according to Eq. (5.17),

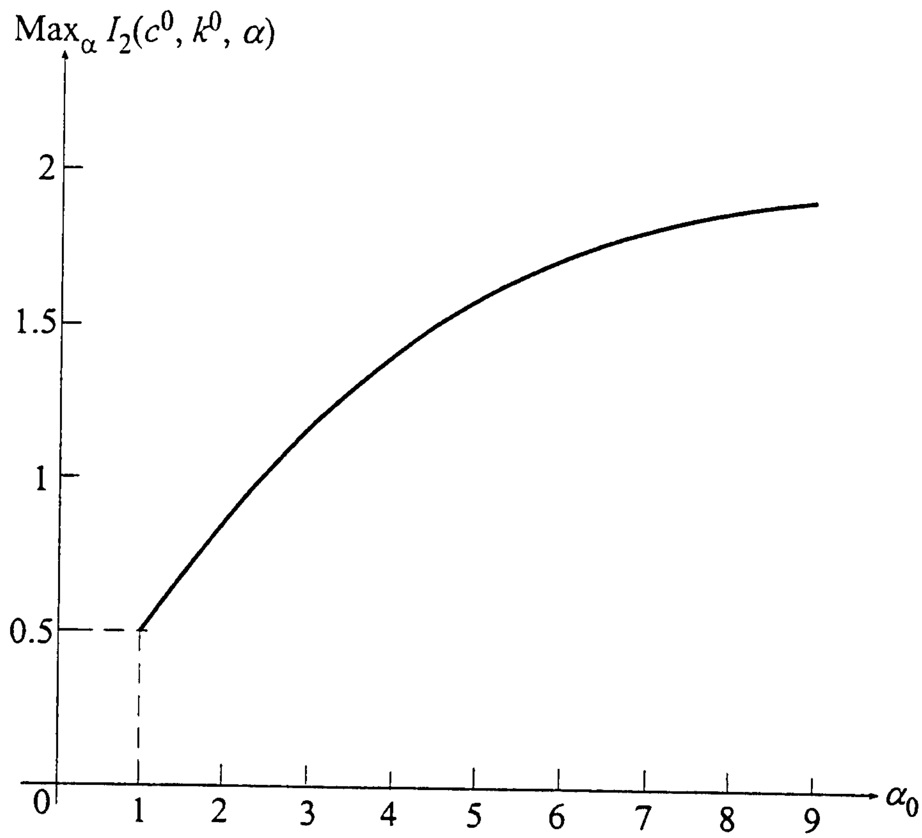


Figure 5-10. Dimensionless optimal value of the maximum of the peak acceleration of the body as a function of the parameter α_0.

$$\alpha_0 = \frac{m_2}{m_1}, \tag{5.100}$$

where m_1 and m_2 are, respectively, the lower and upper bounds for possible values of the mass of the body to be isolated, represented in dimensional variables. As follows from Eq. (5.100) the case of large α_0 corresponds to either $m_2 \to \infty$ or $m_1 \to 0$. In other words, the larger the uncertainty in the knowledge of the mass of the body, the greater the value of the parameter α_0.

The asymptotic relations of Eqs. (5.97) to (5.99) can appear to be useful for the practical calculation of the optimal parameters. The asymptotic relations relieve the designer of the necessity to construct numerically the curves shown in Figs. 5.8 to 5.10 for large α_0. To find the value of α_0 beyond which the curves need not be calculated, one may proceed in the following manner. Plot the curves according to the closed-form relations of Eqs. (5.97) to (5.99). Then plot numerically the curves specified by Eqs. (5.91) to (5.93), proceeding from $\alpha_0 = 1$. The numerical construction can be halted when the numerical curves come sufficiently close to the asymptotic curves.

Finally, represent the asymptotic relations of Eqs. (5.91) to (5.93) in the original dimensional variables. The dimensional and the dimensionless variables are related by Eq. (5.17), in which we set $n = 1$ and $r = 2$, since the isolator with a linear spring and a quadratic damper is being considered. The asymptotic relations in the dimensional variables have the form

$$c^0 \approx \frac{1}{D}\sqrt{\frac{m_1 m_2}{2}}, \quad \frac{m_2}{m_1} \gg 1, \tag{5.101}$$

$$k^0 \approx m_2 \frac{\beta^2}{D^2}, \quad \frac{m_2}{m_1} \gg 1, \tag{5.102}$$

$$\max_{m \in [m_1, m_2]} I_2(c^0, k^0, m) \approx \frac{\beta^2}{D}\sqrt{\frac{m_2}{2m_1}}, \quad \frac{m_2}{m_1} \gg 1. \tag{5.103}$$

Equations (5.101) to (5.103) provide the asymptotic solution of Problem 5.2. To obtain the solution to Problem 5.1, one may use the relations of Eq. (5.64) to (5.67), where $(c^0)', (k^0)'$, and I_2' correspond to the solution of Problem 5.2 in the dimensionless variables.

5.4 CONCLUSIONS.

In this chapter, we have investigated problems of the optimization of characteristics of isolators protecting a body from an instantaneous impact when the mass of the body is not precisely prescribed. Let us summarize the basic results.

1. For isolators with the power law characteristics of Eq. (5.2), with the exponents n and r satisfying the conditions $n \geq 1$ and $r \geq 2$, the optimal damping and stiffness coefficients lie on the boundary of the admissible domain. The admissible domain is specified by Eq. (5.7) for the problem of minimization of the maximum displacement, and by Eq. (5.9) for the problem of minimization of the maximum acceleration.

2. An algorithm to calculate the optimal parameters has been outlined in Section 5.2.3.

3. The complete solution to the problem of the choice of the optimal damping and stiffness coefficients for the isolator consisting of a linear spring and a damper with quadratic characteristic has been constructed in Section 5.3. This solution is represented by the plots in Figs. 5.8 to 5.10 and the asymptotic relations of Eqs. (5.97) to (5.99).

4. For the system with a linear spring and a damper with a quadratic characteristic, the maximum displacement occurs for the maximum of the possible values of the mass of the body being isolated, whereas the maximum of the peak acceleration occurs for the minimum value of the mass.

SECTION 6

IMPLEMENTATION OF OPTIMAL SHOCK ISOLATION SYSTEMS

6.1 CLASSIFICATION OF SHOCK ISOLATORS ACCORDING TO TYPES OF CONTROLLERS.

The isolation systems used in practice can be classified into passive isolation systems, active isolation systems, and adjustable isolation systems.

Passive isolators that contain elastic and damping members and do not use energy sources and automatic control systems are most commonly used. The major advantage of these isolators is relatively low cost and operating reliability.

Active shock and vibration isolation systems require a power supply as well as sensors and controllers to form the control force to be applied to the object to be protected. Currently, active isolation systems have been used in Russia mostly for the protection of devices and equipment from low-frequency vibrations. Active systems for shock isolation are relatively rare in practice. Possibly, a reason for this is that active shock isolation systems would require a powerful energy supply and high-speed (low response time) controller.

Adjustable isolators use passive elements (e.g., stiffness or damping elements) that can be adjusted (tuned or regulated) to the disturbance when operating. For example, isolation systems with adjustable hydraulic dampers are controlled by a throttle valve which regulates the conditions of flow of the damping liquid. The essential difference of adjustable systems from active ones is that in active systems energy is spent directly for the control of the object to be protected, whereas in adjustable isolators energy is spent for the control of a tuning device for passive elements. For the hydraulic damper, for example, the tuning device is the throttle valve. The control of the tuning device require much less energy than the direct control of the object to be protected.

It is possible, of course, to utilize more than one of the above isolator types in a single isolation system.

The design of the isolation system is critically dependent on the excitation of the base for the object to be protected. For example, this base could undergo translation or rotation or could perform a more complicated motion.

6.2 DESIGN OF SHOCK ISOLATION SYSTEMS.

The major task of the design of shock isolation systems for machines or devices is to provide an effective protection from shock excitations and at the same time to ensure that the peak displacement of the object to be protected relative to the base lies within admissible limits.

The design process can be divided into four stages:

1. Analysis of characteristics of the shock disturbances on the basis of the statistical processing of data of full-scale tests and modeling. Mathematical description of the shock disturbance in the form of a set of deterministic excitations or in the form of a stochastic process.

2. Determination of maximum loads allowable for the object to be protected and admissible overall dimensions of the system. At this stage, the data of the mathematical simulation and bench testing of the system to be design are used.

3. Research and Development:

The choice of the type of the isolation system (active, passive, adjustable, or combined).

The choice of the design schematic of the system.

Construction of the mathematical model of the system and the determination of the design variables.

4. Creation of the prototype of the system and testing in various operating conditions.

Let us discuss in more detail the Research and Development substages.

The type of the isolation system is determined with allowance for complexity, reliability, mass, and cost of the system, as well as technological capabilities for manufacturing of systems of this kind.

The selection of the design schematic of the isolation system involves the selection of the design and type of the units of the system, in particular, elastic and damping members. The designer makes a final decision after having compared various available units according to the complexity, size, mass, cost, and other criteria.

The construction of the mathematical model and calculation of the design variables of the isolator, as a rule, consists of two stages:

- Creation of a simplified mathematical model for the analytical investigation of the behavior of the system to be designed and preliminary determination of the design variables.
- Development of the complete mathematical model, computer simulation of the system, and the final determination of the design variables.

The natural tendency of designers to make the system more compact and to reduce the force transmitted to the object to be isolated leads to the necessity to use methods of optimization in designing shock isolation systems.

The statement of the basic optimization problems and methods for their solution were discussed in previous chapters. It should be noted that practical implementation of optimal shock isolation systems designed according to the solution of optimization problems encounters substantial difficulties. Let us outline some sources of these difficulties.

1. Currently, optimal feedback controls have been constructed only for relatively simple systems with one degree of freedom. Actual systems, as a rule, are more complex. It should be noted that optimal open-loop controls have been constructed (in particular,numerically) also for multi-body systems with many degrees of freedom. But practical implementation normally involves a feedback system. The synthesis of optimal feedback controls for systems with many degrees of freedom is a challenge.

2. The designed optimal isolators are optimal only for certain disturbances and can be nonoptimal for other ones. Actual systems, as a rule, are subject to unpredictable disturbances that are very difficult to allow for in designing the optimal isolator. Thus, almost inevitably, the theoretically optimal isolator will turn out to be nonoptimal for some of the disturbances.

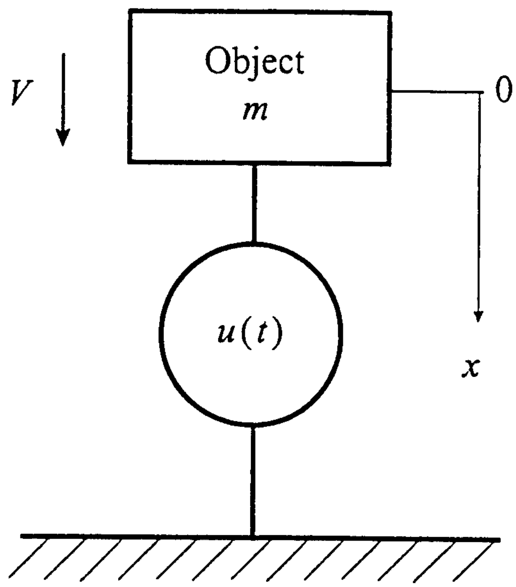
On the other hand, it would be a mistake to consider that the mathematical theory of optimal design is fully abstract and cannot be applied in design practice. Of great importance for practice is the analysis of the isolation system to be designed for the limiting capabilities (limiting performance). The limiting performance analysis involving the solution of optimal control problems allows one to evaluate a lower bound for the performance index of shock isolation, for example, the peak displacement of the body to be isolated or the peak force transmitted to it. It is appropriate to underline here that, although the limiting performance analysis, as a rule, yields an optimal control in an open-loop (feedforward) form, it is not the optimal control that is most valuable. More important is the lower bound of the performance index corresponding to the optimal control. Having the lower bound for the isolation performance index available, the designer then can construct a feedback controller approximating the lower bound of the performance index of the system. At this stage, various methods of parametric optimization can be applied.

6.3 IMPLEMENTATION OF OPTIMAL SHOCK ISOLATORS.

Consider now designs of optimal shock isolators implemented in the former Soviet Union. We will concentrate on passive isolators. So far, active and adjustable isolators have not been used in practical systems of shock isolation. Only experimental investigations have been carried out.

As shown in Chapter 2, for a single-degree-of-freedom system subject to an impulse, the optimal isolator minimizing the peak force transmitted to the body to be protected under the constraint on its peak displacement must act with a constant force until the peak displacement is reached. Recall the problem of limiting capabilities of isolation of a single-degree-of-freedom system from an impulsive impact (Fig. 6.1).

The velocity V can be interpreted either as the velocity at which the object is approaching a fixed base or as the relative velocity of the body with respect to the base resulting from an impulsive loading applied to the base. The former interpretation is relevant for problems of isolation of an object from the shock due to an impact on a fixed solid surface. This is the case, for example, for the design of aircraft landing gears or railroad car shock absorbers. The latter interpretation is relevant for problems of protection of occupants or devices inside a moving vehicle subject to an impact. This is the case, for example, for problems of injury protection of occupants of an automobile in crash situations.



x is the coordinate of the object relative to the base;
V is the initial velocity of the body relative to the base;
$-u$ is the control force transmitted by the isolator to the object

Figure 6-1. Single-degree-of-freedom system.

Equation of motion:

$$m\ddot{x} + u(t) = 0. \tag{6.1}$$

Initial condition:

$$x(0) = 0, \quad \dot{x}(0) = V. \tag{6.2}$$

Performance criteria:

$J_1(u) = \max_t |x| -$ peak displacement;

$J_2(u) = \max_t |u| -$ peak force transmitted to the body.

6.3.0.1 Problem: Find an open-loop control $u_0(t)$ such that

$$J_2(u_0) = \min_u J_2(u), \quad J_1(u) \leq D. \tag{6.3}$$

Solution:

$$u_0(t) = \begin{cases} \frac{mV^2}{2D}, & 0 \leq t \leq \frac{2D}{V} \\ 0, & t > \frac{2D}{V} \end{cases} \tag{6.4}$$

$$J_2(u_0) = \frac{mV^2}{2D}, \quad J_1(u_0) = D. \tag{6.5}$$

From the time history of the optimal control, it is apparent that on the time interval from the beginning of the control to the instant at which the velocity of the body vanishes, the isolator acts on the body with a constant force of magnitude $mV^2/(2D)$.

In Chapter 2, it was proven that the constant force isolator provides the limiting performance for the isolation of an object from an impulsive (instantaneous) impact. Also, passive isolators providing the limiting performance of isolation were identified, among which is the Coulomb damper and the isolator with a linear spring and quadratic law damper. The constant force principle underlies the design of a number of shock isolators which are currently in use.

6.3.1 Railroad Car Shock Absorber with a Dry-Friction Damper.

Railroad car shock absorbers are placed at attachment devices of cars and serve for mitigating impacts of cars against one another in the formation of a train.

6.3.1.1 General Schematic of the Shock Absorber. Let us outline the operation of this device. When the rod (pusher) enters the cylinder, it moves the sliders apart and presses these against the

rough internal surface of the cylinder (Fig. 6.2.) This causes friction force between the contact surfaces of the sliders and the cylinder. For the rod moving in the positive direction of the x-axis, the magnitude of the force acting on the rod (by the sliders) is defined as

$$F(x) = P(x)\frac{\sin\beta(\sin\alpha + f\cos\alpha)}{\sin\alpha(\sin\beta - f\cos\beta)} \tag{6.6}$$

where f is the friction coefficient, α and β are the angles defined by the geometrical shape of the sliders, and $P(x)$ is the force due to the deformation of the elastic member.

The elastic member is usually either a spiral spring or a pneumatic spring. The pneumatic spring is a device containing a compressible volume of gas. For such springs, the "elastic" force P is due to the gas pressure inside the volume. Usually, the elastic member is designed so as to make the function $P(x)$ is relatively smooth. See, for example, the plot of $P(x)$ for the pneumatic spring (Fig. 6.3.)

In Fig. 6.3, D is the design stroke of the rod (the length through which the rod is allowed to move into the cylinder), and P_0 is the initial force of the elastic member (for the rod not exerting pressure on the sliders). If the operating stroke of the rod is relatively small, the shock absorber generates a near-constant force.

6.3.1.2 Design Calculation. The design variables of this absorber can be adjusted so as to provide the constant force U_0 to be generated by the optimal isolator for the instantaneous impact. The design variables P_0, α, β, and f must be adjusted to satisfy the relation

$$U = \frac{mV^2}{2D} = P_0\frac{\sin\beta(\sin\alpha + f\cos\alpha)}{\sin\alpha(\sin\beta - f\cos\beta)}. \tag{6.7}$$

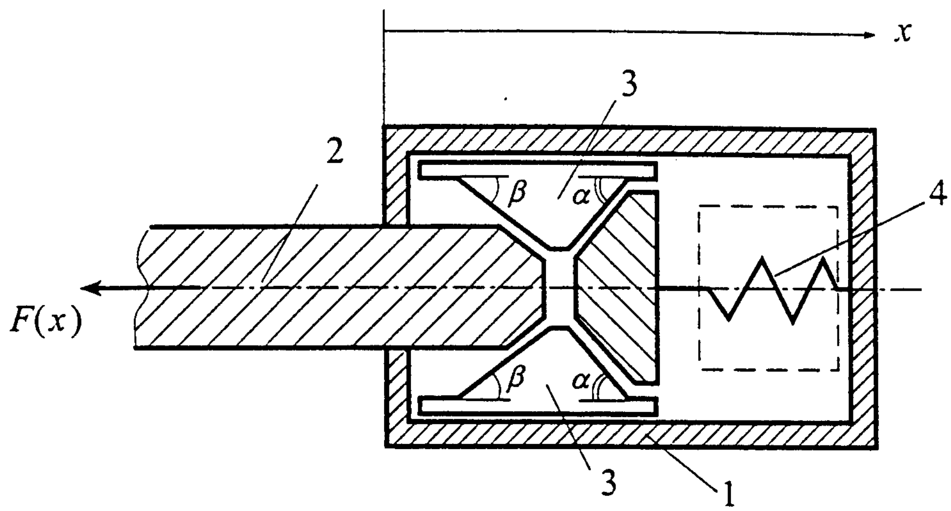
If the design variables differ from the optimal ones, the force generated by the isolator will accordingly differ from the optimal value. To illustrate this, consider a simple example.

Let us analyze the process of attachment of a car to a train. Let m_0 be the mass of the car to be attached to the train, V_0 the velocity at which the car is approaching the train, and D_0 the maximum allowable stroke of the shock absorber. For the optimal absorber designed for these parameters, the design control force is

$$U_0 = \frac{m_0V_0^2}{2D_0}. \tag{6.8}$$

Note that we are considering a passive isolator. Hence, the design variables are fixed and cannot be adjusted during the operating of the device. Suppose now that the actual mass m_1 of the car is different from the design mass m_0. The optimal control force adjusted for the mass m_1, the velocity of the car and the stroke of the absorber being unchanged, would be

$$U_1 = \frac{m_1V_0^2}{2D_0}. \tag{6.9}$$



1- Cylindrical housing

2- Rod

3- Sliders

4-Elastic precompressed member

Figure 6-2. General schematic of railroad car shock absorber.

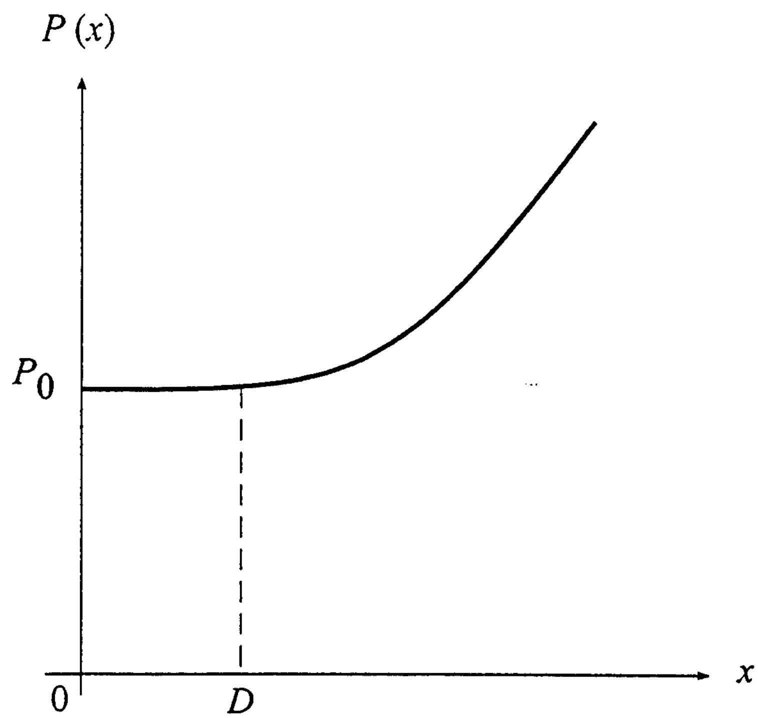


Figure 6-3. Characteristic of pneumatic spring.

However, the absorber has been designed for the mass $m = m_0$ of the car and, hence, although the actual value of m is m_1, the control force will be U_0. The actual peak displacement of the car will be

$$D_1 = \frac{m_1}{m_0} D_0. \tag{6.10}$$

From (6.8), (6.9), and (6.10), it is apparent that

$$U_1 > U_0, \quad D_1 > D_0 \tag{6.11}$$

for $m_1 > m_0$, and

$$U_1 < U_0, \quad D_1 < D_0 \tag{6.12}$$

for $m_1 < m_0$. Hence, for the actual mass exceeding the design mass of the car, the constraint on the maximum displacement will be violated. For the actual mass being less than the design mass, the absorber will operate within the limits of the constraints, but the force U_0 generated by the absorber will exceed the force U_1 optimal for the mass $m = m_1$. Thus, in both cases, the performance of the shock isolation will be different from the optimal performance.

A similar analysis can be carried out for the case where the actual velocity V_1 of impact is different from the design velocity V_0. This yields the relations

$$\frac{\tilde{U}_1}{U_0} = \frac{V_1^2}{V_0^2}, \quad \frac{\tilde{D}_1}{D_0} = \frac{V_1^2}{V_0^2}, \tag{6.13}$$

where U_0 and D_0 are the design values related by (6.8), $\tilde{U}_1$ is the optimal value of the force U corresponding to the velocity V_1, and $\tilde{D}_1$ is the peak displacement of the car equipped with the absorber designed for the velocity V_0 if the actual velocity is V_1. The quantities $\tilde{U}_1$ and $\tilde{D}_1$ are defined as

$$\tilde{U}_1 = \frac{m_0 V_1^2}{2D_0}, \quad \tilde{D}_1 = \frac{m_0 V_1^2}{2U_0}. \tag{6.14}$$

From (6.13) and (6.14) it is apparent that for the car equipped with the shock absorber designed for the impact velocity V_0, we will have

$$\tilde{D}_1 > D_0, \quad \tilde{U}_1 > U_0 \tag{6.15}$$

for $V_1 > V_0$ and

$$\tilde{D}_1 < D_0, \quad \tilde{U}_1 < U_0 \tag{6.16}$$

for $V_1 < V_0$.

In the case of (6.15) the actual displacement of the car will exceed the maximum allowable value D_0. In the case of (6.16), the constraint on the maximum displacement will be satisfied by the actual control force U_0 but the optimal control force $\tilde{U}_1$, corresponding to the velocity V_1, will be exceeded.

These simple considerations demonstrate that a passive optimal isolator provides the optimal performance of isolation only if the actual external disturbance coincides with the design disturbance to which the design variables correspond.

In practice, railroad car shock absorbers are designed for the maximum allowable mass of the car and the maximum allowable impact velocity. This is sometimes referred to as the *worst-case design*. It provides the guaranteed minimum for the performance index. This means that shock absorber so designed demonstrates the optimal performance for the worst disturbance, and for the other possible disturbances all constraints are satisfied and the value of the performance index does not exceed that corresponding to the worst disturbance. For a disturbance other than the worst disturbance, such an absorber is, in general, nonoptimal.

6.3.2 Hydraulic-Pneumatic Shock Isolator for Aircraft Landing Gear.

Landing gear shock absorbers are intended for the reduction of forces transmitted to the aircraft from the impact during landing, as well as from the disturbances due to unevenness of the runway. To be effective, shock absorbers must have appropriate elastic and dissipative characteristics.

6.3.2.1 Design Schematic of Hydraulic-Pneumatic Absorbers. The shock absorber consists of a steel cylinder and a rod which can move in the cylinder in two guide bearings (Fig. 6.4.) One of the guide bearings is rigidly attached to the rod and moves together with it, and the other is attached to the cylinder. Between the inner surface of the cylinder and the outer surface of the rod, in the bottom part of the cylinder, there is a sealing packing. The guide bearing attached to the rod has holes through which the damping liquid can flow into the space between the cylinder and the rod.

Inside the cylinder, there is a tube plunger with holes for the damping liquid to flow through. The plunger is attached to the cylinder at the top part. In the space between the internal surface of the cylinder and the rod, back valves are placed to hamper the flow of the liquid into this space during the backward motion of the rod, thus increasing the damping effect.

6.3.2.2 Operation of the Shock Absorber. If the shock absorber is unloaded (for example, when the aircraft is approaching the landing but has not yet touched down on the runway), the rod is moved as far as possible out of the cylinder and the gas in the pneumatic spring chamber occupies the maximum possible volume. When the external force is applied to the shock absorber (for example, by the wheels of the landing gear), the rod moves into the cylinder, and the damping liquid partially fills the chamber, compressing the gas. This leads to the elastic reaction of the pneumatic spring. As the rod is moving, the liquid flows through the holes in the tube plunger producing the damping effect.

6.3.2.3 Forces Acting on the Cylinder when the Rod Moves into it. When the rod moves in the cylinder, the latter is acted upon by

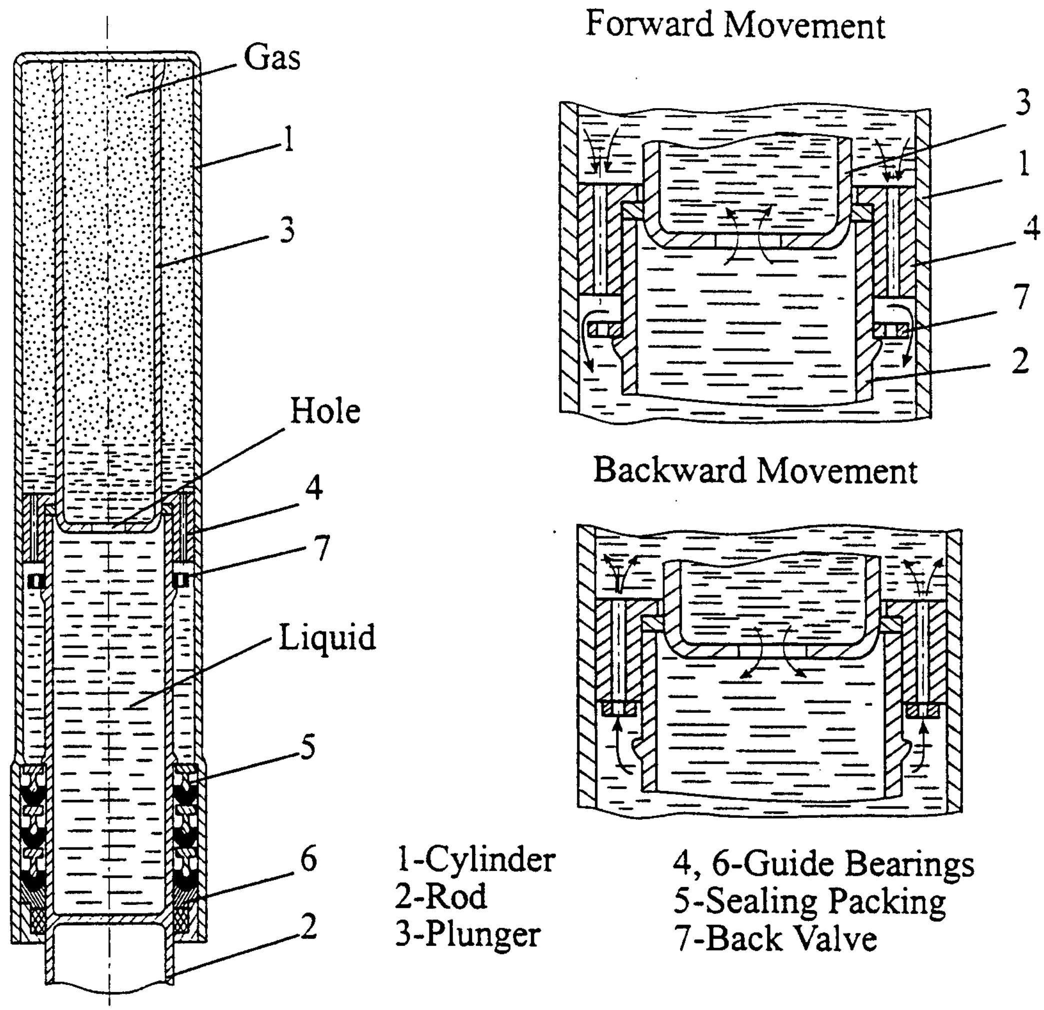


Figure 6-4. Hydraulic-pneumatic shock isolator for aircraft landing gear.

- the force due to the gas pressure, F_g;
- the hydraulic drag force, F_h;
- the force of (dry) friction in the guide bearings, F_f.

The total force acting along the axis of the cylinder is

$$F = F_g + F_h + F_f. \tag{6.17}$$

The major contribution to the total force is that of the components F_g and F_h. The friction force F_f makes up about 10% of F.

Consider the dependence of the forces F_g and F_h on the shock absorber design variables and characteristics of motion of the rod in the cylinder.

6.3.2.4 Gas Pressure Force. We assume the working medium in the pneumatic spring of the absorber to be an ideal gas and the process of compression to be polytropic. By definition, a polytropic process occurs at a constant heat capacitance. In particular, if the heat capacitance is zero, the process is said to be adiabatic. The polytropic process with an ideal gas is characterized by the equation

$$pv^{\chi} = \text{const}, \tag{6.18}$$

where p and v are the pressure and the volume of the gas, respectively, and χ is the polytropic exponent.

Let p_0 and v_0 be the pressure and the volume of the gas, respectively, for the home position of the rod at which it is moved out of the cylinder by the maximum length; A_1 the cross-sectional area of the cylinder corresponding to the inner diameter of it; and x the length by which the rod is moved into the cylinder. Then the current volume of the gas, corresponding to the position x of the rod, is

$$v = v_0 - A_1 x. \tag{6.19}$$

According to (6.18), we have

$$pv^{\chi} = p_0 v_0^{\chi}, \tag{6.20}$$

where p is the current pressure of the gas. Solve (6.20) for p to obtain

$$p = p_0 \left(\frac{v_0}{v}\right)^{\chi}. \tag{6.21}$$

On substituting (6.19) into (6.21), we arrive at the expression

$$p = p_0 \left(\frac{v_0}{v_0 - A_1 x} \right)^{\chi}. \tag{6.22}$$

Multiply (6.22) by A_1 to get the desired force

$$F_g = F_g^0 \left(\frac{1}{1 - x/H_0} \right)^{\chi}, \tag{6.23}$$

where

$$F_g^0 = A_1 p_0, \quad H_0 = \frac{v_0}{A_1}. \tag{6.24}$$

6.3.2.5 Hydraulic Drag Force. Consider a perforated piston (plunger) moving into a cylinder filled with a liquid. In hydraulics, the difference of pressures of the liquid at the inlet and outlet of a channel (hole) is defined as

$$\Delta p = p_+ - p_- = \xi \frac{\rho v_h^2}{2}, \tag{6.25}$$

where p_+ and p_- are the inlet and outlet pressure, respectively, ρ is the density of the liquid. v_h is the average velocity of flow of the liquid through the channel, and ξ is the drag coefficient. The coefficient ξ is determined experimentally. It depends on the viscosity of the liquid and the shape of the channel. As a rule, ξ ranges between 2 and 3.The velocity v_h can be expressed through the velocity $\dot{x}$ of motion of the rod from the continuity equation

$$v_h a = A_2 \dot{x}, \tag{6.26}$$

where a is the total area of the holes in the plunger and A_2 is the working area of the rod. Solve (6.26) for v_h and substitute the result into (6.25) to obtain

$$\Delta p = \xi \frac{\rho}{2} \frac{A_2^2}{a^2} \dot{x}^2. \tag{6.27}$$

To determine the force F_h acting on the plunger, multiply (6.27) by A_2. This yields

$$F_h = \Delta p A_2 = \xi \frac{\rho}{2} \frac{A_2^3}{a^2} \dot{x}^2. \tag{6.28}$$

6.3.2.6 Design Calculation. A simple example can be helpful in studying the process of design of the shock absorber for the aircraft landing gear. Consider a one-leg model of landing shown in Fig. 6.5.

The mass of the landing gear, which is much less than that of the aircraft, is neglected. We also do not take into account the compliance of the wheel tires. This model corresponds to the single-degree-of-freedom system

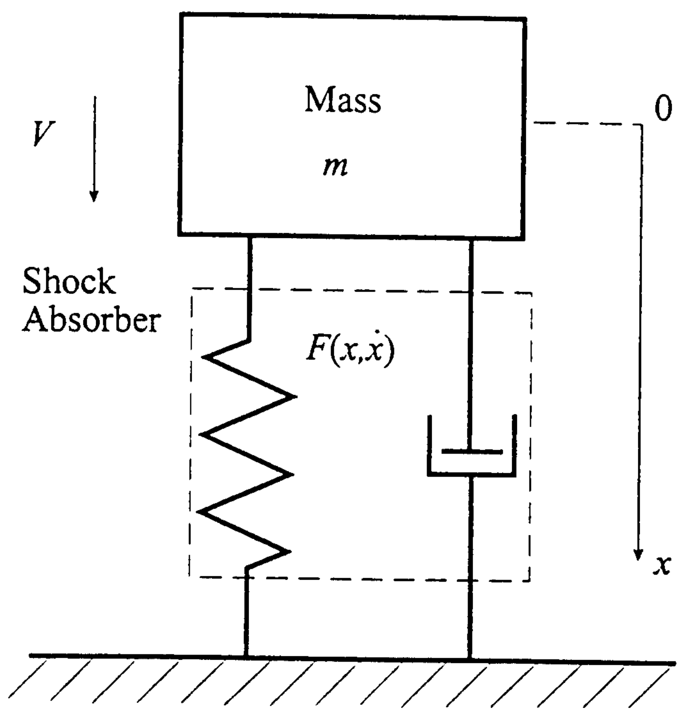


Figure 6-5. One-leg model of landing.

$$m\ddot{x} = -F(x, \dot{x}) + mg(1 - \beta), \tag{6.29}$$

where x is the length by which the rod of the shock absorber is moved into the cylinder (the coordinate of the system); m is the portion of the mass of the aircraft borne by the leg; v_0 is the vertical velocity of landing; $F(x, \dot{x})$ is the magnitude of the force acted on the aircraft by the shock absorber; and β is the lifting force coefficient. Since we are considering landing, we assume the coefficient β to range from 0 to 1. The greater the β, the larger the portion of the weight of the aircraft compensated for by the lifting force. In what follows, we assume, for simplicity, that $\beta = 1$, i.e., the weight of the aircraft is completely counterbalanced by the lifting force.

An essential stage of the design of the aircraft landing gear involves the calculation of the design variables of the shock absorber so as to provide an effective isolation of the aircraft from the impact from landing.

According to the results of the limiting performance analysis, the optimal isolator (shock absorber) must act with the constant force $mV_0^2/(2D)$, where D is the maximum allowable stroke of the rod of the shock absorber (equal to the maximum allowable vertical displacement of the aircraft). Hence, to implement the optimal isolator we should choose the design variables so that the relation

$$F(x(t), \dot{x}(t)) = \frac{mV_0^2}{2D} \tag{6.30}$$

holds.

According to (6.17), the force F is the sum of three components, specifically, the gas pressure force F_g of (6.23), the drag (damping) force F_h of (6.28), and the force of friction in the guide bearings. The friction component is considerably smaller than the other two components and, hence, will be neglected in what follows. With allowance for (6.17), (6.23), and (6.28), the isolator characteristic $F(x, \dot{x})$ can be represented as

$$F(x, \dot{x}) = q\dot{x}^2 + \varphi(x), \tag{6.31}$$

where

$$q = \xi \frac{\rho}{2} \frac{A_2^3}{a^2}, \quad \varphi(x) = F_g^0 \frac{1}{(1 - x/H_0)^\chi}. \tag{6.32}$$

If the parameters q, F_g^0, and H_0 are constant, then it is not possible by selection of these parameters to provide a constant force F. To ensure the constant force, the shock absorber is designed so that the coefficient q depends on x.

Let us find the dependence $q(x)$ providing the constant force ensuring the limiting isolation capabilities. The limiting performance of the isolation system corresponds to the following motion of the object to be protected (the aircraft):

$$x(t) = V_0 t - \frac{V_0^2}{4D} t^2, \quad 0 \le t \le \frac{2D}{V_0}, \tag{6.33}$$

$$\dot{x}(t) = V_0 - \frac{V_0^2}{2D}t, \quad 0 \le t \le \frac{2D}{V_0}. \tag{6.34}$$

The decelerating force

$$F(x(t), \dot{x}(t)) = \frac{mV_0^2}{2D} \tag{6.35}$$

is constant during the optimal motion. During the motion described by the expressions of (6.33) and (6.34), the coordinate x and the velocity $\dot{x}$ are related by

$$\dot{x}^2 = V_0^2(1 - \frac{x}{D}). \tag{6.36}$$

To verify (6.36), place (6.33) and (6.34) into this relation. Substitute (6.35) and (6.36) into (6.31) to obtain

$$\frac{mV_0^2}{2D} = qV_0^2\left(1 - \frac{x}{D}\right) + \varphi(x). \tag{6.37}$$

The solution of (6.37) for q yields

$$q(x) = \frac{m}{2}\frac{1 - \frac{2D}{mV_0^2}\varphi(x)}{D - x}. \tag{6.38}$$

Since the force $q\dot{x}^2$ is a damping force, the function $q(x)$ must be positive for $x < D$. Hence, the numerator of (6.38) must be positive for $x < D$. Note that the denominator of (6.38) vanishes at $x = D$, i.e., at the time instant when the displacement of the rod in the cylinder reaches its maximum value. Hence, to avoid the tendency of $q(x)$ to infinity as $x \to D$, the numerator of the expression of (6.38) must approach zero so that the limit of the ratio remain finite. These two requirements must be satisfied by adjusting the design variables of the pneumatic spring which has the characteristic

$$\varphi(x) = F_g^0\frac{1}{(1 - x/H_0)}, \quad F_g^0 = A_1p_0, \tag{6.39}$$

where F_g^0 and H_0 are defined by (6.24). For the shock absorber under consideration, the coefficient F_g^0 has been chosen to be

$$F_g^0 = \frac{mV_0^2}{2D}\left(1 - \frac{D}{H_0}\right)^{\chi}. \tag{6.40}$$

Note that

$$D < H_0, \tag{6.41}$$

since $H_0 = v_0/A_1$ is the length of the portion of the cylinder filled with the gas when the rod is in the home position. The displacement of the rod cannot exceed this value.

To provide the value of (6.40) for the coefficient F_g^0, an appropriate amount of gas is pumped inside the cylinder. The final expression for the dependence of the damping coefficient of the absorber on the coordinate x has the form

$$q(x) = \frac{m}{2}\frac{[(1 - x/H_0)^\chi - (1 - D/H_0)^\chi]}{(D - x)(1 - x/H_0)^\chi}. \tag{6.42}$$

6.3.2.7 Implementation of the Dependence $q(x)$. According to (6.32), we have

$$q = \xi\frac{\rho}{2}\frac{A_2^3}{a^2}, \tag{6.43}$$

where ρ is the density of the damping liquid, A_2 is the working area of the rod, and a is the total area of the holes of the plunger through which the liquid flows when the rod moves. To provide the dependence $q(x)$ of (6.42), the shock absorber is designed so that the area a varies as the rod moves, the law of variation of a being matched with the function $q(x)$ of (6.42). By virtue of (6.42) and (6.43), we have

$$a(x) = \left[\xi\frac{\rho}{2}\frac{A_2^3}{q(x)}\right]^{1/2}. \tag{6.44}$$

The function $a(x)$ of (6.44) is plotted in Fig. 6.6.

Technically, the variability of the area of the holes through which the damping liquid flows is implemented as follows. A needle of a variable cross-section is attached to the rod (Fig. 6.7.) As the rod moves, the needle moves through a hole (channel) and changes the clearance through which the liquid can flow. By appropriately profiling the needle along its length, we can provide the required dependence for $a(x)$.

6.3.3 Recoil Hydraulic-Pneumatic Isolator for the Barrel of an Artillery Gun.

When an artillery gun is fired, gun powder gases exert pressure on the bottom of the barrel of the gun. To reduce the force transmitted to the gun carriage, the barrel is attached to the gun carriage by means of the recoil absorber, which is a sort of shock isolator. The task of the design of the recoil absorber involves the determination of the design variables of a hydraulic-pneumatic device so as to minimize the force transmitted to the gun carriage, provided the displacement of the barrel does not exceed the maximum allowable value.

6.3.3.1 Design Schematic of the Recoil Absorber. The design of the recoil absorber has much in common with that of the shock absorber of the aircraft landing gear. A design schematic of the recoil brake is shown in Fig. 6.8. When rod 2 moves into the cylinder, the liquid is displaced through channels 4 into chambers IIa and III and compresses the gas. We will refer to this process as the forward movement. It occurs during the barrel recoil. The forward movement is followed by

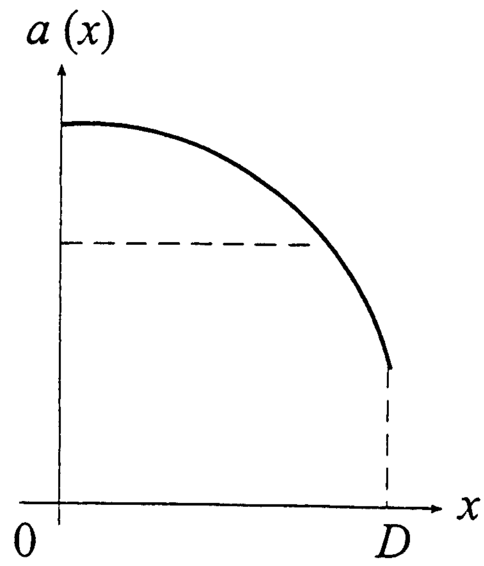


Figure 6-6. Area of hole as the rod moves forward.

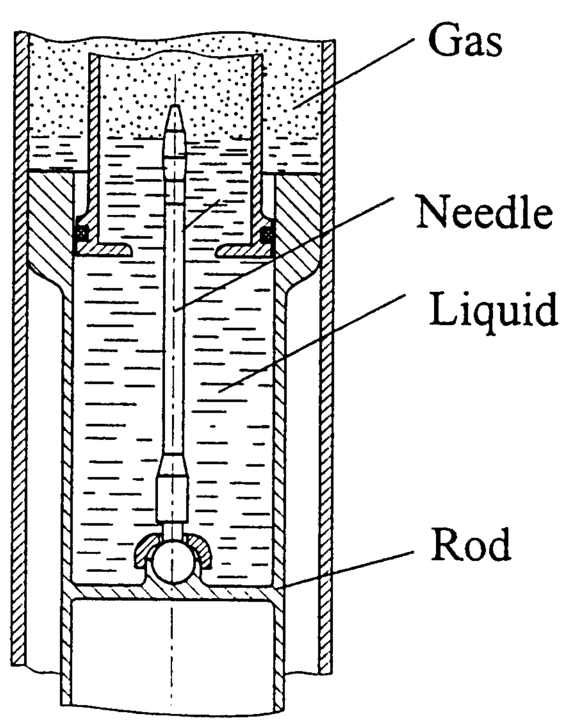
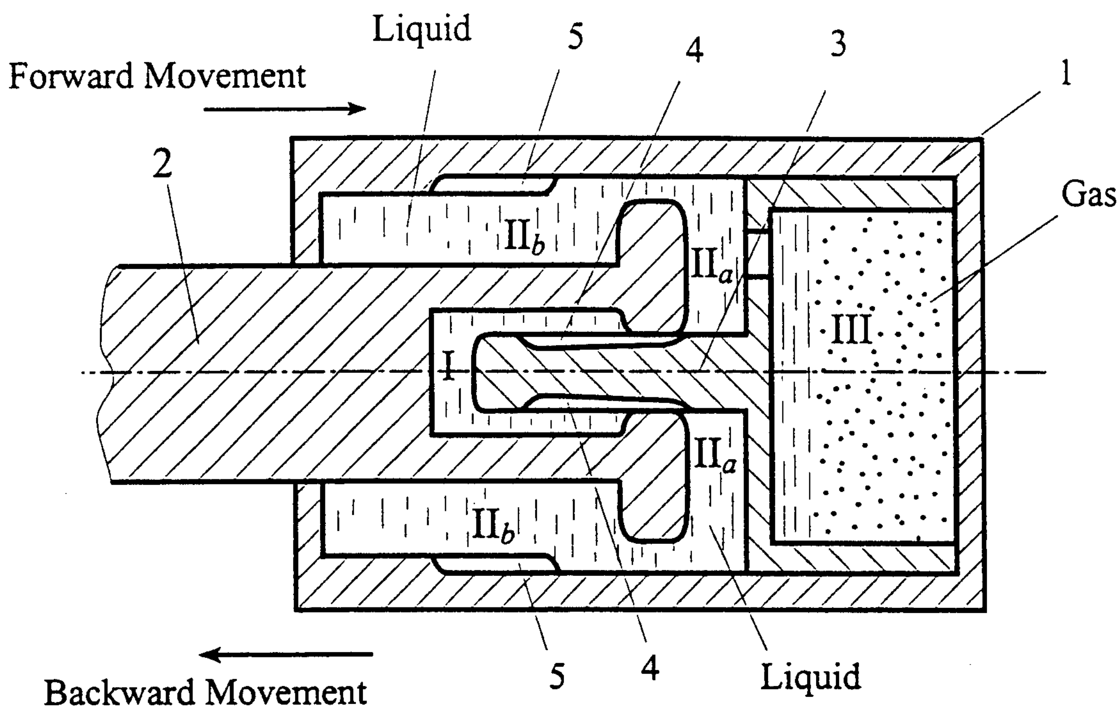


Figure 6-7. Implementation of the Dependence a (x).



1- Cylinder
2- Rod
3- Plunger

4- Channels for flow of damping liquid (forward movement)
5- Channels for flow of damping liquid (backward movement)

Figure 6-8. Recoil hydraulic-pneumatic isolator for the barrel of an artillery gun.

the backward movement during which the gas expands, displacing the liquid from chamber II into chamber IIa and pushing out the rod. During the backward movement, the velocity of the rod is decreased by the hydraulic drag due to the displacement of the liquid from chamber IIb through channels 5 into chambers I and IIa.

In what follows, we will consider only the forward stroke, since it is this stage of motion in which the peak force transmitted to the gun carriage occurs. The design task is to reduce this peak by controlling the cross-sectional area of channels 5 depending on the displacement of the rod in the cylinder.

6.3.3.2 Mathematical Model ofRecoil. The recoil process can be modeled by the single-degree-of-freedom system shown in Fig. 6.9. The system consists of a body which is attached to a fixed base by a hydraulic-pneumatic device and can move along a rough plane inclined at the angle ψ_0 with respect to the horizontal. Coulomb friction acts between the body and the plane. The system has a stop which prevents the body from moving beyond it. When the body is set against the stop, the pneumatic spring acts on the body with a nonzero force pointing to the stop. The body is identified with the barrel of the gun, the base, including the inclined plane, with the gun carriage, and the hydraulic-pneumatic device with the recoil absorber. The angle ψ_0 is the angle of inclination of the gun barrel with respect to the horizontal.

Introduce the coordinate axis pointing downward along the inclined plane. Place the origin of the coordinate system at the position corresponding to the body set against the stop (Fig. 6.9.)

The motion of the system is described by the differential equation

$$m\ddot{x} = -\tilde{R}(x, \dot{x}) + mg\sin\psi_0 + N + R(t) \tag{6.45}$$

with the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0. \tag{6.46}$$

Here, m is the mass of the barrel, $\tilde{R}(x, \dot{x})$ is the magnitude of the force exerted on the barrel by the recoil absorber, g is the acceleration due to gravity, N is the force exerted on the barrel by the gun carriage, and $R(t)$ is the disturbance force due to the firing. The force N is the friction force if the barrel does not have contact with the stop. Otherwise, the reaction force of the stop must be added to the friction force.

The characteristic $\tilde{R}(x, \dot{x})$ of the hydraulic-pneumatic shock absorber has the form

$$\tilde{R}(x, \dot{x}) = \xi \frac{\rho}{2} \frac{A_2^3}{a^2(x)} \dot{x}^2 + \frac{\varphi_0}{(1 - x/H_0)^{\chi}}. \tag{6.47}$$

(See the section devoted to the shock absorber of the aircraft landing gear.) The first term in (6.47) describes the characteristic of the hydraulic damper and second one the characteristic of the pneumatic spring.

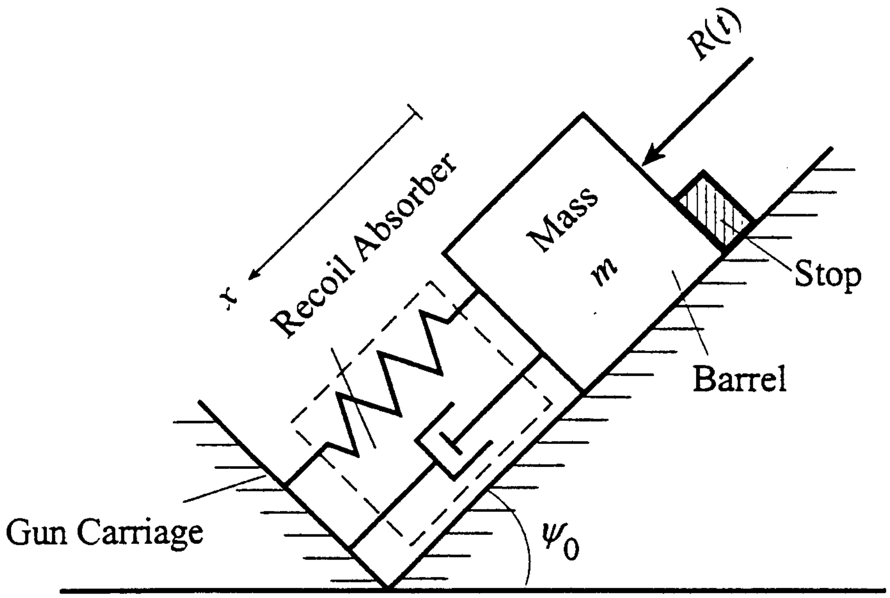


Figure 6-9. Model of recoil.

The time of action of the disturbance $R(t)$ is much less than the time of recoil. Hence, at least at the first stage of the design, it is reasonable to assume that the barrel undergoes an impulsive shock. Then the barrel instantaneously acquires the initial velocity V, and the motion of the system after the shock during the forward movement is described by the equation

$$m\ddot{x} = -R_0(x, \dot{x}) + mg \sin \psi_0 \tag{6.48}$$

with the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = V. \tag{6.49}$$

The quantity R_0 in (6.48) is defined as

$$R_0(x, \dot{x}) = \tilde{R}(x, \dot{x}) + \mu mg \cos \psi_0. \tag{6.50}$$

The first tern in (6.50) is the isolator characteristic of (6.47) and the second term is the Coulomb friction force; μ is the friction coefficient. The quantity $R_0(x, \dot{x})$ is the force transmitted to the gun carriage.

The design task involves the determination of the law $a(x)$ of variation of the area of the channels for the damping liquid flow, depending on the position of the barrel. The function $a(x)$ can be constructed so as to provide the minimum peak force transmitted to the gun carriage, satisfying at the same time the constraint on the displacement of the barrel.

The solution of this problem does not differ essentially from that of the problem of the optimal design of the shock absorber for the aircraft landing gear. The desired function $a(x)$ is constructed so as to ensure that the force transmitted to the gun carriage is constant throughout the forward movement and corresponds to the limiting performance characteristic. Accordingly, the quantity $R_0(x, \dot{x})$ of (6.50) must be a constant defined as

$$R_0(x, \dot{x}) \equiv \frac{mV^2}{2D} + mg \sin \psi_0. \tag{6.51}$$

Denote the right-hand side of (6.51) by U :

$$U = \frac{mV^2}{2D} + mg \sin \psi_0. \tag{6.52}$$

Solve the equation of (6.50) for $\tilde{R}$ to obtain

$$\tilde{R}(x, \dot{x}) = R_0(x, \dot{x}) - \mu mg \cos \psi_0 = \frac{mV^2}{2D} + mg(\sin \psi_0 - \mu \cos \psi_0). \tag{6.53}$$

Equate the right-hand side of (6.53) to the right-hand side of (6.47) and then solve the resulting equation for a to obtain

$$a(x) = \left[\frac{\xi \rho A^3}{2}\right]^{1/2} \left[\frac{D - x}{mV^2/(2D) + mg(\sin\psi_0 - \mu\cos\psi_0) - \varphi(x)}\right]^{1/2}, \tag{6.54}$$

where

$$\varphi(x) = \frac{\varphi_0}{(1 - x/H_0)^{\chi}}.$$

The function $\varphi(x)$ is the characteristic of the pneumatic spring.

SECTION 7

REFERENCES

Afimiwala, K. A., and Mayne, R. W., 1974, "Optimum Design of an Impact Absorber," *J. of Engineering for Industry*, Vol. 96, No. 1 (UNCLASSIFIED).

Aizerman, M. A., 1966, *Theory of Automatic Regulation*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Akulenko, L. D., and Bolotnik, N. N., 1979, "On the Vibration Isolation of Rotating Parts of Mechanisms," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 4, pp.40-47 (UNCLASSIFIED).

Akulenko, L. D., Bolotnik, N. N., and Kaplunov, A. A., 1982, "Isolation of Rotating Parts of Mechanisms," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 1, pp (UNCLASSIFIED).

Andronov, A. A., Vitt, A. A., and Khaikin, S. E., 1959, *Theory of Oscillations*, (in Russian), Fizmatgiz, Moscow (UNCLASSIFIED).

Babitskii, V. I., and Izrailovich, M. Ya., 1968, "On a Problem of Optimal Isolation," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp. 44 (UNCLASSIFIED).

Balandin, D. V., 1985, "Parametric Optimization of Nonlinear Isolators," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 3, pp. 72-74 (UNCLASSIFIED).

Balandin, D. V., 1988, "Optimization of Shock Isolators for Incompletely Prescribed Mass of the Object to Be Protected," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 3, pp.27-31 (UNCLASSIFIED).

Balandin, D. V., 1989a, "Optimization of Shock Isolators for a Class of External Disturbances," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 1, pp. 53-60 (UNCLASSIFIED).

Balandin, D. V., 1989b, "Optimal Control of Damping in Shock Isolators," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 6, pp.66-73 (UNCLASSIFIED).

Balandin, D. V., 1993a, "On Maximum Energy of Mechanical Systems under Shock Disturbances," *Shock and Vibration*, Vol. 1, No. 2, pp.135-144 (UNCLASSIFIED).

Balandin, D. V., 1993b, "On the Perturbation Accumulation in Linear and Nonlinear Systems under Shock Disturbances," (in Russian), *Prikladnaya Matematika i Mekhanika (Applied Mathematics and Mechanics)*, Vol. 57, No. 1, pp. 20-25 (UNCLASSIFIED).

Balandin, D. V., 1994a, "Limiting Capabilities of Control of a Linear System," (in Russian), *Doklady RAN*, Vol. 334, No. 5, pp. 571-573 (UNCLASSIFIED).

Balandin, D. V., 1994b, "Limiting Capabilities of Vibration Isolation of a Multi-Mass Elastic Structure," (in Russian), *Izv. RAN. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 6. pp. 10-17 (UNCLASSIFIED).

Balandin, D. V., 1995a, "Optimal Damping of Vibrations in Elastic Objects," (in Russian). *Prikladnaya Matematika i Mekhanika (Applied Mathematics and Mechanics)*. Vol. 59, No. 3. pp. 464-474 (UNCLASSIFIED).

Balandin, D. V., 1995b, "Limiting Vibroisolation Control of an Oscillating String on a Moving Base," Shock and Vibration, Vol. 2, No. 2, pp. 163-171 (UNCLASSIFIED).

Balandin, D. V., and Malov, Yu. Ya., 1987, "Optimization of Parameters of Isolators for Random Shock Disturbances," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 3, pp. 27-33 (UNCLASSIFIED).

Balandin, D. V., and Markov, A. A., 1986, "Optimization of Parameters of Nonlinear Shock Isolators," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 2, pp. 61-66 (UNCLASSIFIED).

Bartel, D. L., and Krauter, A. I., 1971, "Time Domain Optimization of a Vibration Absorber," *Journal for Engineering for Industry,* Trans. ASME, Ser. B, Vol. 93, No. 3, pp. 799-804 (UNCLASSIFIED).

Bellman, R., 1957, "On the Minimum of Maximum Deviation," *Quart. Appl. Math.*, Vol. 14, No. 4, pp. 419-422 (UNCLASSIFIED).

Bellman, R., 1957, *Dynamic Programming*, Princeton Univ. Press, Princeton, NJ (UNCLASSIFIED).

Bellman, R, 1960, *Introduction to Matrix Analysis,* McGraw-Hill Book Company, New York (UNCLASSIFIED).

Bellman, R., and Dreyfus, S., 1962, *Applied Dynamic Programming,* Princeton Press, Princeton, NJ (UNCLASSIFIED).

Bolotin, V. V., 1969, "Theory of Reliability for Mechanical Systems with Finite Number of Degrees of Freedom", (in Russian), *Izv. AN SSSR Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp. 73-81 (UNCLASSIFIED).

Bolotin, V. V., 1970, "Theory of Optimal Protection from Vibration under Stochastic Disturbances", (in Russian), *Trudy Moskovskogo Energeticheskogo Instituta*, Issue 74, pp. 5-15 (UNCLASSIFIED).

Bolotnik, N. N., 1974, "Optimization of Parameters of Some Mechanical Vibratory Systems." (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp. 33-40 (UNCLASSIFIED).

Bolotnik, N. N., 1975, "Optimization of Parameters of a Mechanical Vibratory System with Dry Friction,"(in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp.

56-58 (UNCLASSIFIED).

Bolotnik, N. N., 1976, "Optimal Isolation Problems for Classes of External Disturbances," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 4, pp. 34-41 (UNCLASSIFIED).

Bolotnik, N. N., 1977, "Optimal Isolation of Torsional Vibrations," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp. 51-55 (UNCLASSIFIED).

Bolotnik, N. N., 1983, *Optimization of Shock and Vibration Isolation Systems*, (in Russian), Nauka, Moscow.

Bolotnik, N. N., 1993, "Optimization of Characteristics and Design Variables of Shock Isolation Systems and Vibration Technological Machines," (in Russian), *Izv. RAN. Tekhnicheskaya Kibernetika*, No. 1, pp. 62-67 (UNCLASSIFIED).

Bolotnik, N. N., and Kaplunov, A. A., 1980, "Some Problems of Optimal Control of Rotation of a Rigid Body," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp. 14 (UNCLASSIFIED).

Boltyanskii, V. G., 1968, *Mathematical Methods of Optimal Control*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Bolychevtsev, E. M., 1971, "The Choice of Optimal Isolation Law Under Shock Disturbances," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 5, pp. 51-54 (UNCLASSIFIED).

Bolychevtsev, E. M., 1973, "Synthesis of Optimal Controls Minimizing the Maximum Displacement Under the Action of Disturbances", (in Russian), *Scientific Transactions of the Institute of Mechanics of Moscow State University*, No. 22 (UNCLASSIFIED).

Bolychevtsev, E. M., and Borisov, A. P., 1976, "Shock Isolation in a Linear System," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 2, pp. 55 (UNCLASSIFIED).

Bolychevtsev, E. M., and Lavrovskii, E. K., 1977, "On Constructing a Pareto-Optimal Set for Certain Optimization Problems," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, Vol. 12, No. 6, pp. 33-40 (UNCLASSIFIED).

Bolychevtsev, E. M., Zhiyanov, N. I., and Lavrovskii, E. K., 1975, "Optimization of Parameters of an Oscillatory system Subject to Impulse Disturbances," (in Russian), *Vestnik MGU (Bulletin of Moscow State University)*, Ser. 1, No. 6, pp. 103 (UNCLASSIFIED).

Bond, R. A., 1969, "Landing Gear Performance and Glide Path Angles," South African Mech. Eng., Vol. 48, No. 2, pp.25-32 (UNCLASSIFIED).

Bryson, A. E., and Ho, Y. C., 1975. *Applied Optimal Control*, John Wiley and Sons, New York (UNCLASSIFIED).

Chernousko, F. L., and Banichuk, N. V., 1973, *Variational Problems of Mechanics and Control.*

Numerical Methods, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Chernousko, F. L., and Melikyan, A. A., 1978, *Game Problems of Control and Search* (in Russian), Nauka, Moscow (UNCLASSIFIED).

Demyanov, V. F., and Malozemov, V. N., *Introduction to Minmax*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Den Hartog, J. P., 1930, "Forced Vibrations with Combined Viscous and Coulomb Damping," *Phys. Mag.*, Vol. 9, No. 59, p. 801 (UNCLASSIFIED).

Den Hartog, J. P., 1931, "Forced Vibrations with Combined Coulomb and Viscous Damping," *Trans. ASME. Appl. Mech.*, Vol. 53, No. 9, p. 107 (UNCLASSIFIED).

Eliseev, V. V., and Malinin, L. M., 1990, "On the Shock Isolation of the Shaft-Line Support." (in Russian), *Problemy Mashinostroeniya i Nadezhnosti Mashin*, No. 2, pp. 29-35 (UNCLASSIFIED).

Fedorenko, R. P., 1978, *Approximate Solution of Optimal Control Problems*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Frolov, K. V., (Ed.), 1995, *Vibration in Engineering. Protection from Vibration and Shock.* (in Russian), Vol. 6, Mashinostroenie, Moscow (UNCLASSIFIED).

Frolov, K. V., and Furman, F. A., 1990, *Applied Theory of Vibration Isolation Systems*, Hemisphere Pub. Corp., New York (UNCLASSIFIED).

Furunzhiev, R. I., 1971, *Design of Optimal Vibration Protection Systems*, (in Russian), Vysshaya Shkola, Minsk (UNCLASSIFIED).

Furunzhiev, R. I., 1977, *Automatic Design of Oscillatory Systems*, (in Russian), Vysshaya Shkola, Minsk (UNCLASSIFIED).

Gantmakher, F. R., 1960, *Lectures on Analytical Mechanics*, (in Russian), Fizmatgiz, Moscow (UNCLASSIFIED).

Genkin, M. D., and Ryaboy, V. M., 1988, *Elastic-Inertial Vibration Isolation Systems: Limititing Performance, Optimal Configurations*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Guretskii, V. V., 1965a, "On One Problem of Optimal Control," (in Russian). *Izv. AN SSSR. Mekhanika*, No. 1, pp. 159-162 (UNCLASSIFIED).

Guretskii, V. V., 1965b, "Limiting Capabilities of the Equipment Protection Against Shock Action," (in Russian), *Izv. AN SSSR. Mekhanika*, No. 2, pp. 76-81 (UNCLASSIFIED).

Guretskii, V. V., 1965c, "On the Maximum of the Displacement of an Optimally Isolated Object," (in Russian), *Trudy LPI (Leningradskii Politekhnicheskii Institut)*, No. 252: *Dinamika i Prochnost Mashin*, pp. 16 (UNCLASSIFIED).

Guretskii, V. V., 1966a, "Determination of Optimal Parameters for Shock Isolators," (in Russian),

Trudy LPI (Leningradskii Politekhnicheskii Institut), No. 266: *Mekhanika i Protsessy Upravleniya.* pp. 17-23 (UNCLASSIFIED).

Guretskii, V. V., 1966b, "On the Choice of Optimal Parameters for Characteristics of Shock Isolators," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela. Inzhenernyi Zhurnal (Mechanics of Solids)*, No. 1, pp. 167-170 (UNCLASSIFIED).

Guretskii, V. V., 1968, "On the Number of the Switching Points of the Optimal Control in the Problem of Minimization of the Peak Displacement," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela. Inzhenernyi Zhurnal (Mechanics of Solids)*, No. 1, pp. 26-30 (UNCLASSIFIED).

Guretskii, V. V., 1969a, "On the Problem of Minimizing the Maximum Displacement," (in Russian), *Trudy LPI (Leningradskii Politekhnicheskii Institut)*, No. 307: *Mekhanika i Protsessy Upravleniya. Vychislitel'naya Matematika*, pp. 11-21 (UNCLASSIFIED).

Guretskii, V. V., 1969b, "On the Limiting Isolation Capabilities for Vibration Loads," *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 1, pp. 51-54 (UNCLASSIFIED).

Guretskii, V. V., Kolovskii, M. Z., and Mazin, L. S., 1970, "On the Limiting Capabilitites of Shock Isolation," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 6, pp. 17-22 (UNCLASSIFIED).

Guretskii, V. V., and Mazin, L. S., 1976, "On the Limiting Capabilities of Active Vibration Isolation," (in Russian), *Prikladnaya Mekhanika*, Vol. 12, No. 7, pp. 109 (UNCLASSIFIED).

Harris, C. M., and Crede, C. E., 1996, *Shock and Vibration Handbook*, McGraw-Hill, New York (UNCLASSIFIED).

Haug, E. J., Arora, J. S., 1979, *Applied Optimal Design. Mechanical and Structural Systems*, Wiley-Interscience Publ. New York (UNCLASSIFIED).

Hsiao, M. H., Haug, E. J., and Arora, J. S., 1979, "A State Space Method for Optimal Design of Vibration Isolators," *J. Mech. Des.*, Vol. 101, No. 4, pp. 309 (UNCLASSIFIED).

Hrovat D., and Hubbard, H., 1987, "A Comparison Between Jerk Optimal and Acceleration Optimal Vibration," *J. Sound and Vibration*, Vol. 112, No. 2, pp. 201-208 (UNCLASSIFIED).

Hundal, M. S., 1976, "Impact Absorber with Linear Spring and Quadratic Law Damper," *J. Sound and Vibration*, Vol. 48, No. 2, p. 189 (UNCLASSIFIED).

Hundal, M. S., 1979, "Response of a Base Excited System with Coulomb and Viscous Friction," *J. Sound and Vibration*, Vol. 64, No. 3, p. 371 (UNCLASSIFIED).

Isaacs, R., 1965, *Differential Games*, Wiley and Sons, New York (UNCLASSIFIED).

Ishlinsky, A. Y., 1963, "Mechanics of Gyroscopic Systems", (in Russian), *Izdatel'stvo AN SSSR*, Moscow (UNCLASSIFIED).

Ishlinsky, A. Y., 1987, *Classical Mechanics and Inertia Forces*, (in Russian), Nauka, Moscow

(UNCLASSIFIED).

Johnson, C. D., 1967, Optimal Control with Chebyshev Minimax Performance Index", *J. of Basic Engineering*, Vol. 89, No. 2 (UNCLASSIFIED).

Karlin, S., 1959, *Mathematical Methods and Theory in Games, Programming, and Economics*, Pergamon Press, London-Paris (UNCLASSIFIED).

Karnopp, D. C., and Trikha, A. K., 1969, "Comparative Study of Optimization Techniques for Shock and Vibration Isolation," *J. Eng. Industry*, Vol. 91, No. 4, pp. 1128-1132 (UNCLASSIFIED).

Kolovskii, M. Z., 1966, *Nonlinear Theory of Vibration Protection Systems*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Kolovskii, M. Z., 1976, *Automatic Control of Vibration Protection Systems*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Kononenko, V. O., and Podchasov, N. P., 1973, "On the Optimum Active Damping of Oscillations," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 3, pp. 3-10 (UNCLASSIFIED).

Krasovskii, N. N., 1968, *Theory of Control of Motion*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Krasovskii, N. N., 1970, *Game Problems of Motion Encounter*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Krasovskii, N. N., 1985, *Control of a Dynamic System. The Problem of Minimum of the Guaranteed Result*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Kulagin, V. V., and Prourzin, V. A., 1985, "Optimal Control of the Spatial Motion of a Rigid Body to Be Isolated from Shock," *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids)*, No. 3, pp. 8-15 (UNCLASSIFIED).

Kuznetsov, A. G., and Chernousko, F. L., 1968, "On the Optimal Control Minimizing the Extremum of a Function of the Phase Variables," (in Russian), *Kibernetika*, No. 3, pp. 50-55 (UNCLASSIFIED).

Kwak, B. M., Arora, J. S., and Haug, E. J., 1975, "Optimum Design of Damped Vibration Absorbers over a Finite Frequency Range," *AIAA Journal*, Vol. 13, No. 4, pp. 540 (UNCLASSIFIED).

Larin, R. M., 1969, "Gradient Method for Solving an Approximate Problem of Optimal Isolator Synthesis", (in Russian), *Trudy LPI (Leningradskii Politekhnicheskii Institut)*, No. 307: *Mekhanika i Protsessy Upravleniya. Vychislitel'naya Matematika*, pp. 155-162 (UNCLASSIFIED).

Larin, V. B., 1974, *Statistical Problems of Vibration Protection*, (in Russian), Naukova Dumka, Kiev (UNCLASSIFIED).

Lee, F. B., and Markus, L., 1967, *Foundations of Optimal Control Theory*, John Wiley and Sons, New York (UNCLASSIFIED).

Leitmann, G., 1981, *The Calculus of Variations and Optimal Control*, Plenum Press, New York (UNCLASSIFIED).

Levitan, E. S., 1960, "Forced Oscillation of Spring-Mass System Having Coulomb and Viscous Damping," *J. Acoust. Soc. Amer.*, Vol. 32, No. 10, p. 1226 (UNCLASSIFIED).

Liber, T., and Sevin, E., 1966, "Optimal Shock Isolation Synthesis," *Shock and Vibration Bull.*, No. 35, Pt. 5, pp. 203-215 (UNCLASSIFIED).

Maksimovich, Yu. P., 1970a, "On the Optimal Vibration Isolation," (in Russian), *Mashinovedenie*, No. 4, pp. 13-20 (UNCLASSIFIED).

Maksimovich, Yu. P., 1970b, "On the Attainable Quality of Protection from Periodic Vibration," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids),* No. 5, pp. 23-27 (UNCLASSIFIED).

Manoilenko, V. D., and Rutman, Yu. L., 1974, "An Elastic Analogy for the Optimal Control of an Isolated Object when the Peak Load is Minimized," *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids),* No. 6, pp. 3-11 (UNCLASSIFIED).

Mazin, L. S., 1969, "The Influence of Errors in Determining the Inertial Characteristics of a Rigid Body on the System of its Isolation," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids),* No. 1, pp. 44-51 (UNCLASSIFIED).

Moiseev, N. N., 1975, *Elements of Theory of Optimal Systems*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Moiseev, N. N., Ivanilov, Y. P., and Stolyarova, E. M., 1978, *Methods of Optimization*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Panovko, Y. G., 1977, *Introduction to the Theory of Mechanical Shock*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Pars, L. A., 1979, *A Treatise on Analytical Dynamics*, Oxbow Press, Woodbridge, Connecticut (UNCLASSIFIED).

Pilkey, W. D., 1968, "Optimization of Shock Isolation Systems," *Transactions, Society of Automotive Engineers*, pp.2900-2909 (UNCLASSIFIED).

Pilkey, W. D., 1970, "Interactive Optimal Design of Isolation Systems," *Shock and Vibration Bulletin,* Vol. 41 (UNCLASSIFIED).

Pilkey, W. D., 1971, "PERFORM – A User-Oriented Computer Programm for the Limiting Performance of Dynamic," *Shock and Vibration Bulletin,* Vol. 42 (UNCLASSIFIED).

Pilkey, W. D., 1973, "Optimal Shock Absorbers in Freight Cars," *ASME Rail Transportation*

Transaction, ASME, 73-RT-3 (UNCLASSIFIED).

Pilkey, W. D., and Kalinovskii, A. J., 1972a, "Response Bounds for Structures with Incompletely Prescribed Loading," *Shock and Vibration Bulletin,* Vol. 43, pp. 31-42 (UNCLASSIFIED).

Pilkey, W. D., and Kalinovskii, A. J., 1972b, "Identification of Shock and Vibration Forces," *Identification of Vibrating Structures, ASME,* pp. 73-86 (UNCLASSIFIED).

Pilkey, W. D., and Kalinovskii, A. J., 1975, "Design for Incompletely Prescribed Loading." *ASCE Engineering Mechanics Journal,* Vol. 101, No. EM4, pp. 505-510 (UNCLASSIFIED).

Pilkey, W. D., and Kitis, L., 1986, "Limiting Performance of Shock Isolation Systems by a Modal Approach," *Earthquake Engineering and Structural Dynamics,* Vol. 14, pp. 75-81 (UNCLASSIFIED).

Pilkey, W. D., and Lim, T. W., 1987, "Optimum Shock Isolation with Minimum Setting Time," *Shock and Vibration Bulletin,* Vol. 57, pp. 379-388 (UNCLASSIFIED).

Pilkey, W. D., and Rosenstein, M., 1988, "Shock Spectra for Classes of Excitations," *Shock and Vibration Bulletin,* Vol. 59, pp. 171-183 (UNCLASSIFIED).

Pilkey, W. D., Sevin, E., and Kalinovskii, A. J., 1968, "Computer Aided Design of Optimum Shock Isolation Systems," *Shock and Vibration Bulletin,* Vol. 39, pp.185-198 (UNCLASSIFIED).

Pilkey, W. D., and Strenkowski, J., 1974, "Optimal Performance of Crashing Automobiles," *Vehicle System Dynamics,* Swets and Zeitinger (UNCLASSIFIED).

Pilkey, W. D., and Wang, B. P., 1972, "Limiting Performance of Ground Transportation Vehicles," *Proc. 19th AIAA/ASME, SAE Structural Dynamic Conf (UNCLASSIFIED).*

Polak, E., 1971, *Computational Methods in Optimization. A Unified Approach.* Academic Press, New York (UNCLASSIFIED).

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., 1962, *Mathematical Theory of Optimal Processes*, Wiley-Interscience, New York (UNCLASSIFIED).

Prourzin, V. A., 1988, "Optimum Shock Isolation of a Round Device," (in Russian), *Izv. AN SSSR. Mekhanika Tverdogo Tela (Mechanics of Solids),* No. 2, pp. 32-37 (UNCLASSIFIED).

Rao, S. S., and Hati, S. K., 1980, "Optimum Design of Shock and Vibration Isolation Systems Using Game Theory," *Eng. Optimization*, Vol. 4, No. 4, pp. 215-226 (UNCLASSIFIED).

Roitenberg, Y. N., 1978, *Automatic Control*, (in Russian), Nauka, Moscow (UNCLASSIFIED).

Rozonoer, L. I., 1959, "Maximum Principle of L. S. Pontryagin in the Theory of Optimal Systems," (in Russian), *Avtomatika i Telemekhanika*, No. 10,11 (UNCLASSIFIED).

Ruzicka, J. E., 1970a, "Passive Shock Isolation," Pt. I, *Sound and Vibration*, Vol. 4. No. 8. pp. 14-24 (UNCLASSIFIED).

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